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Investigation of the Mechanism of High Energy Nucleon-Nucleus Interaction.

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Summary. — Nuclear interaction of 9 GeV protons with emulsion nuclei have been investigated. A composite nuclear interaction was built up from the individual events and the angular and energy distribution of identified shower particles were measured. From these data the mean value of the inelasticity coefficient as well as the mean number of nucleons struck in a proton-nucleus collision was calculated for an « average » emulsion nucleus. It was shown by comparing the present experimental results with those found for nucleon-nucleon collision at 6.2 GeV ([15,16]) that the nucleon-nucleus interaction is better described at these energies by the cascade model than by the tunnel model. From this the mean life of the excited state of a colliding nucleon-nucleon system was estimated to be less than $3 \cdot 10^{-23}$ s.

1. – Introduction.

Several authors have investigated the nuclear interactions of high energy accelerated protons in emulsions and compared the experimental results with the predictions of the cascade and tunnel models ([1-3]). The investigation of

([1]) G. B. ŽDANOV, V. M. MAKSIMENKO, M. I. TRETIAKOVA and M. N. TSHERBAKOVA: *Žurn. Èksp. Teór. Fiz.*, **37**, 620 (1959).

([2]) V. S. BARASHENKOV, V. A. BELIAKOV, V. V. GLAGOLEV, N. DALKHAZHAV, YAO TSYNG SE, F. KIRILLOVA, R. M. LEBEDEV, V. M. MALTSEV, P. K. MARKOV, M. G. SHAFRANOVA, K. D. TOLSTOV, E. I. TSIGANOV and WANG SHOU FENG: *Nucl. Phys.*, **14**, 522 (1959).

([3]) E. M. FRIEDLÄNDER: *Nuovo Cimento*, **14**, 796 (1959).

internuclear processes in nucleon-nucleus (or nucleus-nucleus) collisions is very important not only for explaining the experimental results of these complex phenomena but for obtaining in this way a direct estimate of the lifetime of the excited state of colliding systems.

Assuming a cascade mechanism the nucleon-nucleus interaction consists of a series of successive elementary interactions of the incident nucleon and its secondaries with the nucleons of the nucleus. Strictly speaking an internuclear cascade can develop only if the time (t_e) between the beginning of the first (nucleon-nucleon) collision and that of the emission of π-mesons from the colliding system is less than the time (t_p) necessary for the colliding system to penetrate the nucleus, the latter depending on the size of the hit nucleus and the impact parameter (*).

If, however, $t_e > (t_p)_{\max}$, where $(t_p)_{\max}$ corresponds to the longest possible path length in the nucleus, the development of an internuclear cascade is impossible and the incident nucleon interacts only with the nucleons along its path as a whole (tube or tunnel model). In this way a tube is cut out of the nucleus by the collision and the colliding system begins to emit π-mesons only after being separated from the remaining nucleus.

Let us define t_e as the mean life of the excited state of the colliding system and measure it in the center-of-mass system (c.m.s.) of the colliding partners (*e.g.* nucleon-tube). The value of t_p in the c.m.s. will be energy dependent due to the Lorentz contraction of the colliding nucleus. Thus testing the validity of the above models at different energies makes possible, at least in principle, to estimate upper and lower limits for the value of t_e.

Since the average number of nucleons inside the nucleus hit directly by the incident nucleon is relatively small also for heavy emulsion nuclei (Ag, Br), the value of t_e estimated on the basis of emulsion measurements can be taken in first approximation also as a measure of the mean life of the excited state of a colliding nucleon-nucleon system.

(*) If two-center models ([4-1]) are adopted for describing nucleon-nucleon or nucleon-nucleus collisions, the treatment becomes more complicated, because two excited centers are moving through the nucleus. Taking into account that the method to be described later gives only a crude estimation of the mean life of the excited state of colliding systems, we shall restrict our considerations to the simpler one center model.

([4]) S. TAKAGI: *Prog. Theor. Phys.*, **7**, 123 (1952).

([5]) W. L. KRAUSHAAR and L. J. MARKS: *Phys. Rev.*, **93**, 326 (1954).

([6]) P. CIOK, T. COGHEN, J. GIERULA, R. HOŁYNSKI, A. JUDEK, M. MIĘSOWICZ, T. SANIEWSKA and J. PERNEGR: *Nuovo Cimento*, **10**, 741 (1958).

([7]) G. COCCONI: *Phys. Rev.*, **111**, 1699 (1958).

([8]) J. BUEMEISTER, K. LANIUS and H. W. MEIER: *Nuovo Cimento*, **11**, 12 (1959).

([9]) K. NIU: *Nuovo Cimento*, **10**, 994 (1958).

([10]) F. J. M. FARLEY: *Nuovo Cimento*, **16**, 209 (1960).

([11]) U. MAOR and G. YEKUTIELI: *Nuovo Cimento*, **17**, 45 (1960).

The comparison of experimental results with those obtained by the cascade or tunnel models is, however, very difficult, because the majority of quantitative predictions of both models are very uncertain in consequence of the lack of a complete and experimentally checked theory of multiple meson production. We shall therefore analyse our experimental results obtained for the nuclear interactions of 9 GeV protons with emulsion nuclei in a rather phenomenological way and compare them with the results of nucleon-nucleon collision measurements to get some information on the mechanism of internuclear processes.

2. – Experimental.

A NIKFI - R emulsion stack was irradiated by the internal proton beam of the proton synchrotron of the Joint Institute of Nuclear Research, Dubna. The stack consisted of emulsion pellicles of 10 cm × 10 cm size and 400 μm thickness.

98 nuclear interactions of the 9 GeV protons found by scanning along the track were measured. The angles of all secondary particles generated in the interaction were measured directly on each track. The mass and momentum of the particles were determined by a combination of grain density and multiple scattering measurements on tracks having projected lengths more than 1 mm per plate. All tracks having an ionization value less than $4J_0$, where J_0 is the ionization of primary protons, were included in the analysis.

Particles were identified to be either π-mesons or protons up to about 1 GeV energy. By calculating the average multiplicity of pions and protons as well as their momentum and transversal momentum distributions and other quantities derived from these, a geometrical correction was applied for losses of particles having projected lengths smaller than 1 mm/plate.

So as to obtain a better statistics a composite nuclear interaction was built up from the measured events; this corresponds, if properly normalized, to a nuclear interaction of a 9 GeV proton with an « average » nucleus of the emulsion. The velocity of the c.m.s. in the laboratory system (l.s.) was determined from [12]

$$\beta_c = \frac{\sum_i p_i \cos \vartheta_i}{\sum_i \gamma_i}, \tag{1}$$

where p_i, γ_i and ϑ_i are the momenta, energies and angles of shower particles in the l.s.

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[12] I. Lord, I. Feinberg and M. Schein: *Phys. Rev.*, **80**, 970 (1950).

Since particles having energies higher than 1 GeV could not be identified, only the upper and lower limits of β_c and $\gamma_c = (1-\beta_c^2)^{-\frac{1}{2}}$ could be determined corresponding to the cases, where all unidentified particles were assumed to be either protons or pions:

$$(1.35 \pm 0.06) \leqslant \gamma_c \leqslant (1.52 \pm 0.07)\,. \tag{2}$$

The angles and momenta of shower particles were transformed into the c.m.s. defined by the limiting values of γ_c. It was found that, transforming and using the lower limit of γ_c, the distribution of π-mesons was nearly symmetrical in the forward and backward cone ($n_f/n_b = 0.88 \pm 0.07$), whereas using the upper limit of γ_c the backward cone was strongly preferred ($n_f/n_b = 0.60 \pm 0.05$). Assuming that pion emission is symmetrical in the c.m.s. of the colliding system the lower limit of γ_c seems to be more probable (Table I). The same argumentation shows that the majority of unidentified particles consists of protons.

TABLE I.

Average multiplicity of shower tracks ($J \leqslant 1.4\, J_0$)	all particles	3.50 ± 0.28
	charged pions	2.73 ± 0.26
	protons	1.08 ± 0.15
Average multiplicity of gray and black tracks		12.0 ± 0.35
Value of γ_c		1.35 ± 0.06
Mean number of nucleons struck inside the emulsion nuclei		5.15 ± 0.85
Mean momentum value (in the c.m.s.) of	pions	230 ± 23
	protons	780 ± 80
Mean transversal momentum of	pions	160 ± 16
	protons	460 ± 46
Mean value of the inelasticity coefficient		0.28 ± 0.05

The angular and momentum distributions transformed in the c.m.s. using the latter value of γ_c are plotted in Fig. 1-4, unidentified particles being considered protons.

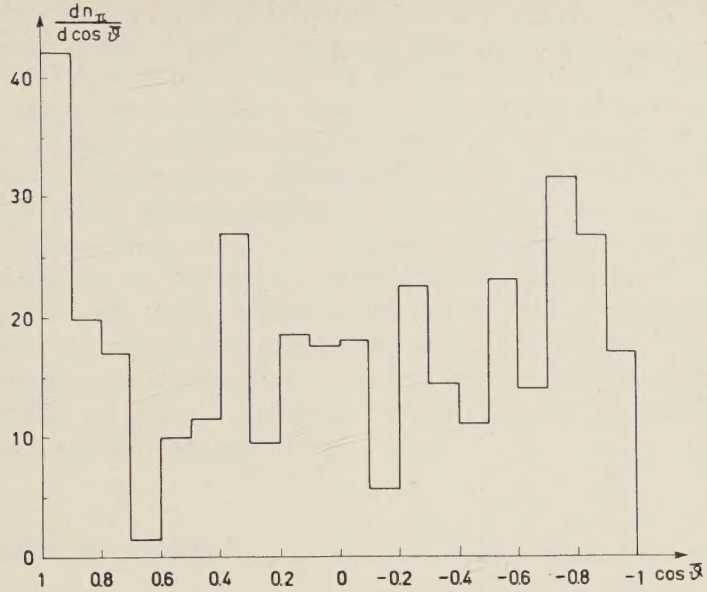

Fig. 1. – Angular distribution of π-mesons in the c.m.s.

To avoid, however, errors due to the uncertainty in the identification of particles the mean value of the momenta and transversal momenta of pions

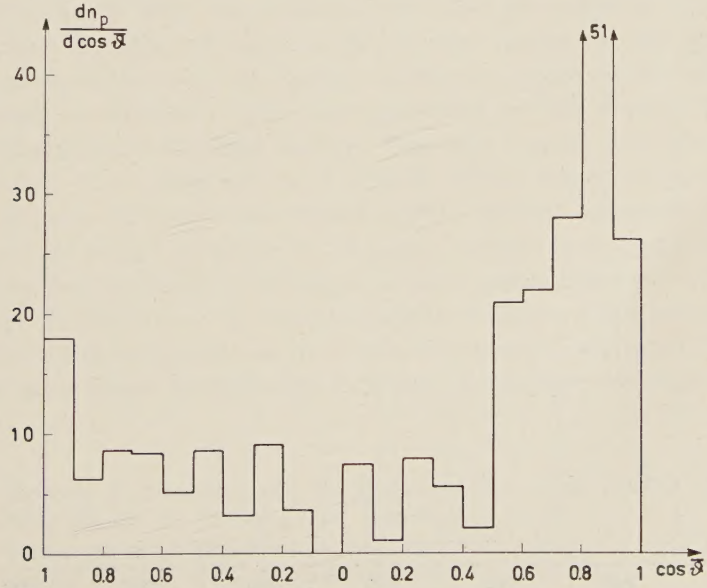

Fig. 2. – Angular distribution of protons in the c.m.s.

and protons given in Table I were calculated only from the data of the backward cone, where all particles were identified. There it was assumed that the

Fig. 3. – Momentum distribution of π-mesons and protons in the c.m.s.

momentum and the transversal momentum distributions are symmetrical in the forward and backward directions in the c.m.s. The mean value of the

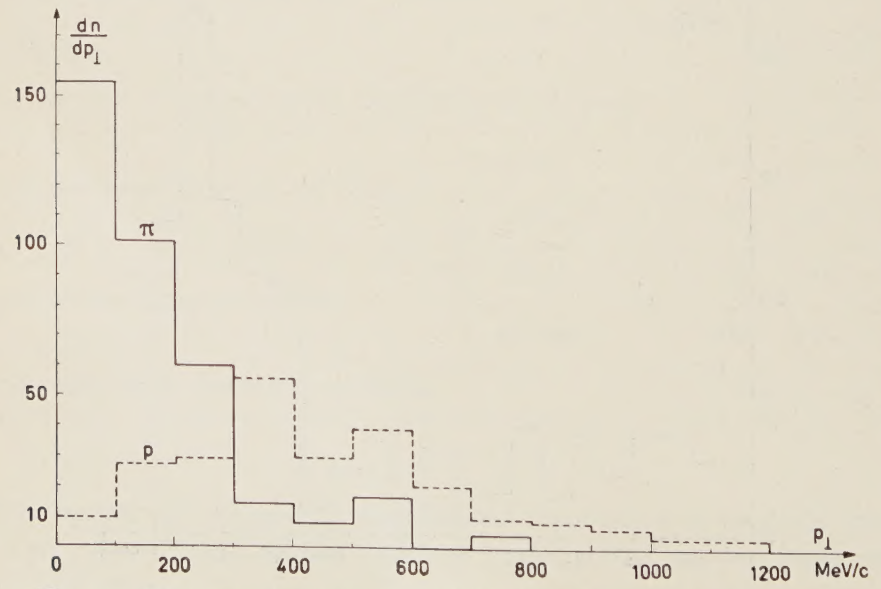

Fig. 4. – Transversal momentum distribution of π-mesons and protons.

inelasticity coefficient was calculated also from the energy distribution of pions emitted in the backward cone under the same assumption.

From the known energy of primary protons and the probable value of the Lorentz factor of the c.m.s. ($\gamma_c = 1.35$) the mean number of nucleons (η) struck inside the « average » emulsion nucleus could be determined ([13]); its value given in Table I is somewhat—although not significantly—higher than the expected value of the average number of nucleons in a tube for an « average » emulsion nucleus. The values of η determined for different groups of events selected according to the number of evaporation tracks show a significant correlation with the mean number of protons emitted. Similar results were found earlier for pion-nucleus interaction at 4.2 GeV ([14]) (*).

The experimental results given in Table I are in relatively good agreement with the results of similar experiments, where all secondary particles (shower and grey tracks) were analysed ([1,2]).

3. – Discussion and conclusion.

As was discussed in the introduction the mean life of the excited state of a colliding nucleon-nucleon system can be determined at least in first approximation by deciding, whether at given energies the experimental data obtained for nucleon-nucleus collisions better fit the predictions of the cascade model or those of the tunnel model.

Different physical parameters may be used in principle to compare the experimental results with the predictions based on the above models. It seems that the most sensitive of them is the average energy or the energy distribution of pions in the c.m.s. If cascade mechanism is realized the emitted pions are scattered or secondary pions are generated in the nuclear matter surrounding the primary colliding system; thus the average pion energy will be less than the value predicted for a tunnel model, where the overwhelming majority of pions will leave the excited system without secondary interaction.

The average energy of pions in the c.m.s. emitted from a colliding nucleon-tube system was assumed to be equal in first approximation to the average pion energy in the c.m.s. in nucleon-nucleon collision, when the energy densities of the colliding systems in their respective c.m.s.'s are equal. The energy density of an average nucleon-tube collision in emulsion (average number of

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([13]) G. Bozóki, E. Fenyves and E. Gombosi: *Rep. of the Central Research Inst. for Phys. of the Hung. Ac. of Science*, **6**, 351 (1958).

([14]) P. Abrahamson, J. Ben-Arieh and G. Yekutieli: *Nuovo Cimento*, **12**, 27 (1959).

(*) A detailed analysis of the frequency distribution of the values of η as compared with the predictions of a simple cascade model will be published elsewhere.

nucleons in the tube ≈ 4) for 9 GeV protons is equal to the energy density for a nucleon-nucleon collision at about 5.3 GeV. Since the nearest energy for which nucleon-nucleon collision results are available in emulsion is 6.2 GeV ([15,16]), these experimental results were used for comparision.

The weighted mean of the average pion energy in the c.m.s. found in these measurements is (0.37 ± 0.02) GeV, *i.e.* significantly higher than (0.27 ± 0.02) GeV, the value found in the present experiment for nucleon-nucleus collisions. This shows according to our argumentation that the collision of 9 GeV protons with emulsion nuclei can be better described by a cascade mechanism than by a tunnel model (*).

This means that the mean life t_e of the excited state of a colliding nucleon-nucleon system must be less than $(t_p)_{\max}^{\mathrm{Ag,Br}} = 3 \cdot 10^{-23}$ s, which corresponds to the diameter of a heavy emulsion nucleus (Ag or Br). Thus we can conclude in first approximation that the mean life of an excited state is

$$t_e \leqslant 3 \cdot 10^{-23} \text{ s} . \tag{3}$$

To obtain the lower limit or a possible lower value for the upper limit of t_e investigations of the nucleon-nucleus collision must be carried out at higher energies.

In connection with this we should like to make the following remark. The relative frequency of high energy interactions appearing to be nucleon-nucleon collisions, *i.e.* having only a few or no evaporation tracks, increases strongly at energies above 10^{12} eV ([17,18]). This phenomenon was explained by EDWARDS *et al.* ([17]) by assuming an increase of the transparency of nuclei at these energies. The above phenomenon can be explained, however, also by assuming that at these energies the nucleon-nucleus collision can be better described by a tunnel model than by a cascade one. Assuming $E = 2 \cdot 10^{12}$ eV/nucleon as limiting energy above which cascade mechanism cannot be valid for an

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([15]) R. M. KALBACH, J. J. LORD and C. H. TSAO: *Phys. Rev.*, **113**, 330 (1959).

([16]) R. R. DANIEL, N. KAMESWARA RAO, P. K. MALHOTRA and Y. TSUZUKI: *Nuovo Cimento*, **16**, 1 (1960).

(*) Similar conclusions were drawn by BARASHENKOV *et al.* ([2]) analysing the nuclear interactions of 9 GeV protons in emulsion, whereas FRIEDLÄNDER ([3]) found a better agreement with a nucleon-tube model. The experimental material used by FRIEDLÄNDER was, however, selected using the criterion $n_s \geqslant 4$, resulting in a very strong bias, the average multiplicity of the unselected events being $\langle n_s \rangle = 3.50 \div 0.28$ (Table I). The influence of this bias on the experimental results and the conclusions drawn from them cannot be estimated directly.

([17]) B. EDWARD, J. LOSTY, D. H. PERKINS, K. PINKAU and J. REYNOLDS: *Phil. Mag.*, **3**, 237 (1958).

([18]) J. PERNEGR: private communication.

average emulsion nucleus, we obtain the lower limit of t_c as about 10^{-24} s, a value being probably not very different from the real value of t_c.

It might be of interest to mention that a wave travelling with light velocity passes during this time a distance of about 0.3 fermi, which is of the order of the radius of the nucleon or the nucleon core ([19]).

* * *

The authors are very much indebted to the Joint Institute for Nuclear Research, Dubna, for the exposed and processed plates.

([19]) G. Domokos: *Reports of the Central Research Inst. for Phys. of Hung. Ac. of Science*, **7**, 385 (1959).

RIASSUNTO (*)

Si sono studiate le interazioni nucleari dei protoni di 9 GeV con i nuclei dell'emulsione. Dai singoli eventi si è costruita una reazione nucleare composita e si sono misurate la distribuzione angolare e la distribuzione dell'energia delle particelle dello sciame identificate. Da questi dati si è calcolato il valore medio del coefficiente di anelasticità ed il numero medio dei nucleoni urtati in una collisione protone-nucleo per un nucleo « medio » dell'emulsione. Si è dimostrato, confrontando i presenti risultati sperimentali con quelli trovati per le collisioni nucleone-nucleone di 6.2 GeV ([15,16]), che l'interazione nucleone-nucleo è meglio descritta a queste energie dal modello a cascata anzichè dal modello a tunnel. Da questo si è stimato che la vita media dello stato eccitato di un sistema nucleone-nucleone in collisione è inferiore a $3 \cdot 10^{-23}$ s.

(*) *Traduzione a cura della Redazione.*

On Recombination Processes in Neutron-Irradiated n-Type Germanium.

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(ricevuto il 29 Dicembre 1960)

Summary. — A calculation of minority charge carriers lifetime in neutron irradiated n-type germanium is performed taking into account the presence of damage regions produced by fast secondaries. The results are compared with experimental data.

1. – Introduction.

Recently Crawford and Gossick [1,2] have proposed a theory of bombardment effects with neutrons in germanium which accounts for the regions of highly concentrated damage that fast secondaries may produce. In n-type germanium these regions would be p-type as this is the conductivity type to which germanium turns as a consequence of irradiation. The created p-regions are supposed to be spherical of radius r_1 and are surrounded by a potential barrier which extends through a radius r_2 outside the p-region.

We would call attention on a possible use of this theory to interpret experimental data on the behaviour of minority charge carriers lifetime in neutron-irradiated n-type germanium. The reduction of lifetime is usually attributed to the introduction, with bombardment, of isolated defects which act as traps for minority carriers [3,4].

This interpretation does not take into account either possible defects of

(1) B. R. Gossick: *Journ. Appl. Phys.*, **30**, 1214 (1959).
(2) J. H. Crawford jr. and J. W. Cleland: *Journ. Appl. Phys.*, **30**, 1204 (1959).
(3) G. K. Wertheim: *Journ. Appl. Phys.*, **30**, 1166 (1959).
(4) O. L. Curtis jr.: *Journ. Appl. Phys.*, **30**, 1174 (1959).

complex structure or the possibility of their aggregation. In n-type germanium, in fact, besides traps originated from isolated defects one should consider the presence of a number of p-regions produced by fast secondaries in the n-matrix.

2. – Theory.

According to a very simplified model the rate of trapping of electrons from the conduction band $R_c^{(n)}$ would be proportional to the electron concentration n in the matrix and to the number $(1 - F)$ of empty traps (created by isolated defects), F being the fraction of traps occupied by electrons.

Moreover some electron would penetrate the p-region. This number could be put proportional to:

1) the electron concentration in n matrix;

2) the fraction F_r of electrons which have sufficient energy to penetrate the potential barrier;

3) the number N_r of regions per volume unity;

4) the cross-section of the region which we would suppose to be the geometrical one multiplied by a suitable correction factor f which accounts for eventual recombination of carriers before they had the possibility to interact with the regions: $\pi r_{\text{eff}}^2 f$ (where for the radius of the region we take the effective one $r_{\text{eff}} = (r_1 r_2)^{\frac{1}{2}}$);

5) the mean velocity of electrons $\bar{v}_n$.

Therefore:

$$R_c^{(n)} = c_n n(1 - F) + nF_r N_r \pi r_{\text{eff}}^2 f \bar{v}_n .$$

The emission rate $R_e^{(n)}$ could be considered also as a sum of two terms: the rate of emission from single traps, proportional to F, and a term due to the passage of electrons from the p-region, where they are present with the concentration n', to the n-region. This term is put proportional to:

1) the electron concentration in the p-region n';

2) the number of empty places left in the n-region by electrons which went over the barrier in the p-region: $(1 - F_r)$;

3) N_r;

4) $f\pi r_{\text{eff}}^2$;

5) the mean velocity of electrons in the p-region $\bar{v}_n'$;

Therefore:

$$R_e^{(n)} = c_n' F + n'(1 - F_r) N_r \pi r_{\text{eff}}^2 f \bar{v}_n' .$$

For holes one would obtain analogue expressions. We now admit complete independence of the two processes controlled by single traps and regions of damage. This is a crude approximation which we hold for lack of a better hypothesis: however the factor f, introduced in the capture cross-section, should, to a certain extent, correct for this.

Performing calculations in the usual manner, with the assumptions that the variations from the equilibrium concentration of electrons and holes in the n-region are small and equal, and that carrier concentrations in the p-region practically remain unalterated, one can get, for the case of n-type germanium, the final result:

$$\frac{1}{\tau} = c_p \left(\frac{n_0}{n_0 + n_1} \right) + \frac{n_0}{c_{rp}^{-1} n_0 + c_{rn}^{-1} p_0'} , \tag{1}$$

where: $c_p = N \sigma_p \bar{v}_p$,

$c_{rp} = f \pi r_{\text{eff}}^2 N_r \bar{v}_p$,

$c_{rn} = f \pi r_{\text{eff}}^2 N_r \bar{v}_n$,

and N is the number of isolated defects created,

σ_p is the hole capture cross-section for isolated defects,

n_0 is the electron concentration at equilibrium in the n-region,

p_0' is the hole concentration at equilibrium in the p-region,

n_1 is the electron concentration when the Fermi level is at the position of the energy level of the single traps.

3. – Comparison with experimental results.

The above expression will now be compared with experimental results of CURTIS, CLELAND and CRAWFORD ([4-7]).

As it is known these authors have interpreted their results on the basis of single defects production. For this case only the first term of eq. (1) is used. The authors found good agreement with experimental data assuming

([5]) O. L. CURTIS jr., J. W. CLELAND, J. H. CRAWFORD jr. and J. C. PIGG: *Journ. Appl. Phys.*, **28**, 1161 (1957).

([6]) O. L. CURTIS jr., J. W. CLELAND and J. H. CRAWFORD jr.: *Journ. Appl. Phys.*, **29**, 1722 (1958).

([7]) O. L. CURTIS jr. and J. W. CLELAND: *Journ. Appl. Phys.*, **31**, 423 (1960).

at room temperature

for ^{60}Co γ-rays $\quad E_c - E_t = 0.20$ eV , $\quad \bar{v}_p \sigma_p = 2 \cdot 10^{-8}$ cm^3/s ,

for fission neutrons $\quad E_c - E_t = 0.20$ eV , $\quad \bar{v}_p \sigma_p = 1 \cdot 10^{-7}$ cm^3/s .

E_t being the energy of the trap level. In our approximation it seems reasonable to admit that the traps associated with single defects although produced by neutrons, are of the same type of those which are found with electron and γ-ray irradiation. We know in fact that these radiations produce the simplest type of damage with single point defects. We therefore use for $\bar{v}_p \sigma_p$ a value in accord with the data with electron and γ-ray irradiation. For example $\bar{v}_p \sigma_p = 2 \cdot 10^{-8}$ cm^3/s at room temperature. The energy level associated with single traps may be put $\simeq 0.20$ eV below the bottom of the conduction band.

We consider now sample AI-I irradiated by CURTIS, CLELAND and CRAWFORD ([5]). They give:

$$n_0 = 4.6 \cdot 10^{14} \text{ electrons/cm}^3 \quad \text{and} \quad n_1 = 6 \cdot 10^{15} \text{ electrons/cm}^3 .$$

In the p-regions created by irradiation the Fermi level will be supposed to be in the asymptotic position after irradiation:

$$E_F = E_v + 0.12 \text{ eV} ,$$

with:

$$p_0' = 2.2 \cdot 10^{17} \text{ holes/cm}^3 .$$

Let us call ψ_p the difference between the Fermi level in the n- and p-type regions; it is

$$\psi_p = 0.27 \text{ eV} .$$

The calculations by CRAWFORD and CLELAND ([2]) lead to conclude that the number of single defects produced by the neutron flux φ used in their experiments is about $0.6\,\varphi$ while the region number is given by $N_r = 0.07\,\varphi$. From their data we can put $r_1 = 120$ Å. In the considered sample we find, by application of the theory ([1,2]):

$$r_2 \simeq 2\,600 \text{ Å} \quad \text{and} \quad r_{\text{eff}}^2 \simeq 3.12 \cdot 10^{-11} \text{ cm} .$$

If we take for the hole mean velocity the value ([7]):

$$\bar{v}_p = 1.8 \cdot 10^7 \text{ cm/s} , .$$

we can calculate the value of the correction factor f which fits the experimental value of $1/\tau_{\text{exp}} = 1.6 \cdot 10^4$ s^{-1} at room temperature.

We find from eq. (1) that with $f = 0.9 \cdot 10^{-1}$ we have $1/\tau_{\text{cal}} = 1.54 \cdot 10^4$ s^{-1}.

Analogous calculations for Crawford and Cleland's sample C-I ([5]) lead, with $f = 0.9 \cdot 10^{-1}$, to $1/\tau_{cal} = 5.1 \cdot 10^4\ s^{-1}$ against the experimental value $1/\tau_{exp} = 5.6 \cdot 10^4\ s^{-1}$ as is given by CLELAND and CRAWFORD.

More interesting results may be obtained by comparison with data obtained by CURTIS and CLELAND ([7]) upon irradiation with monoenergetic 14 MeV neutrons.

In this case we could expect the damage to consist exclusively in the production of damage regions with a cross-section equal to the fast neutrons scattering cross-section (about $2 \cdot 10^{-24}\ cm^2$).

Eq. (1) reduces to

$$\frac{1}{\tau} = \frac{c_{rp} n_0}{n_0 + (c_{rp}/c_{rn}) p_0'} . \tag{2}$$

Authors' sample A4-I has resistivity $\varrho = 2\ \Omega\,cm$, $n_0 = 8.7 \cdot 10^{14}$ electrons/cm^3. When irradiated with $\varphi = 2.2 \cdot 10^{11}$ neutrons/cm^2 one has: $\psi_p \simeq 0.26\ eV$, $p_0' = 2.2 \cdot 10^{17}$ holes/cm^3, $N_r = 0.09\,\varphi$.

In this case, by application of the Holmes and Leibfried ([8]) theory corrected for ionization losses one has $r_1 \simeq 860\ Å$. We then find $r_2 \simeq 4300\ Å$ and $r_{eff}^2 \simeq 3.6 \cdot 10^{-10}\ cm$. Eq. (2), using the same f factor than above, gives $1/\tau_{cal} \simeq 0.9 \cdot 10^5\ s^{-1}$ against about $1 \cdot 10^5\ s^{-1}$ as inferred from the authors' curves.

In this case the authors were able to find accord between the single defect formula and experience using a capture cross-section σ_p comparable with the value found in pile irradiation but an energy level for recombination located 0.32 eV above the valence band. This value is in evident contrast with pile cases.

4. – Conclusion.

Neutron damage in n-type germanium consists of the production of single defects as well as of damage regions. In the minority charge carriers lifetime calculation the presence of these damage regions has to be taken into account. The theoretical results presented allow for this and are in good agreement with experimental data at room temperature.

([8]) D. K. HOLMES and G. LEIBFRIED: *Journ. Appl. Phys.*, **31**, 1046 (1960).

RIASSUNTO

È presentato un calcolo del tempo di vita medio dei portatori di cariche in minoranza nel germanio di tipo n irradiato con neutroni che tiene conto della presenza di regioni di danno prodotte da secondari veloci. I risultati sono poi confrontati con i dati sperimentali.

Stopping Power of C for ^{210}Po α-Particles (*).

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(ricevuto il 9 Gennaio 1961)

Summary. — The range of ^{210}Po α-particles has been measured in pure elemental carbon. The carbon foils were prepared by spraying colloidal graphite in isopropyl alcohol onto a glass plate, drying, and then floating on the surface of water. Semi-conductor counters were used as detectors. The range was found to be 4.43 mg/cm^2, which corresponds to a value of 0.899 ± 0.009 for the atomic stopping power of carbon relative to air. This agrees well with earlier values obtained from measurements with compounds by use of Bragg's additive law.

1. – Introduction.

The range-energy relations for α-particles in most materials are quite well established [1]. However in the case of carbon there is some uncertainty owing to the fact that elemental carbon occurs in nature in forms that are difficult to make into foils. For this reason the stopping power of pure carbon has been obtained indirectly heretofore by applying Bragg's additive law to measurements on such gaseous compounds as CO, CO_2, CH_4, C_2H_4, etc. A measurement of the stopping power of elemental carbon, made by Yaramis [2] in 1953, gives a value which appears to be at least 33% higher than the accepted value based on the Bragg law. In view of this result, it seemed worth while to make another direct measurement of the stopping power of C for ^{210}Po α-particles.

(*) Work performed under the auspices of the U. S. Atomic Energy Commission.

(**) On leave from the University of Istanbul.

(1) H. A. Bethe and J. Ashkin: in *Experimental Nuclear Physics*, edited by E. Segrè, vol. 1 (New York, 1953), p. 166.

(2) B. Yaramis: *Rev. Fac. Sci. Univ. Istanbul*, **18**, 369 (1953).

2. – Experimental set-up.

The experimental arrangement consisted of a vacuum chamber (Fig. 1), a semiconductor detector, preamplifier, amplifier, scaler, oscilloscope, and

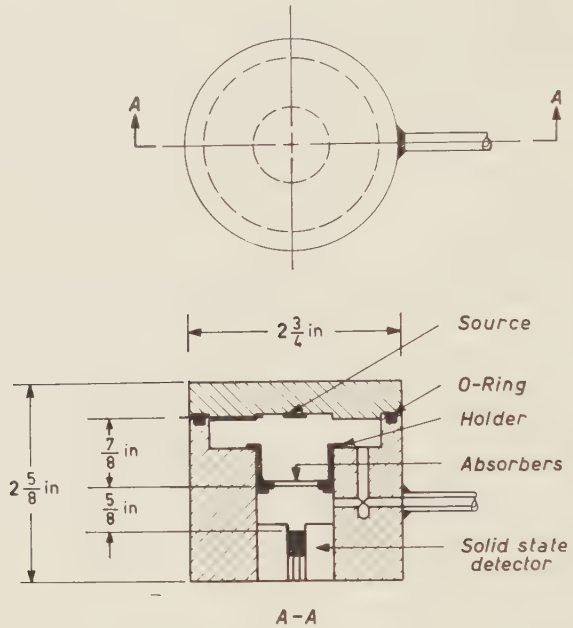

Fig. 1. – Vacuum chamber with source, absorbers, and detector. Two absorbers are shown in position.

200-channel analyser. The detector was an RCA n-p silicon junction counter, the bias connection of which is given in Fig. 2. It gave a pulse height that varied linearly with energy. The resolution of the counter was studied as a function of the counter bias for a given amplification. The optimum bias, defined as that which gave the best energy resolution, was found to be -16.5 V with approximately 1.2% energy spread for 5.3 MeV α-particles.

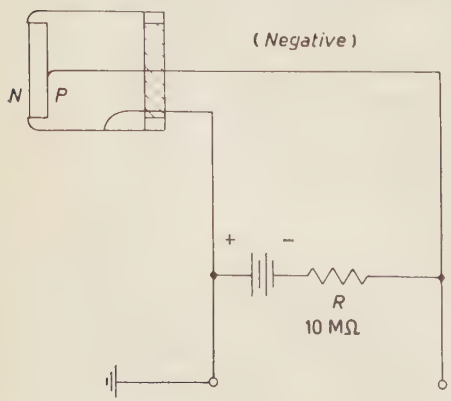

Fig. 2. – Bias connection of the RCA detector. $V = -16.5$ V.

The carbon was prepared by using Dag dispersion No. 154, which is a form of colloidal graphite in isopropyl alcohol. This product, which has good film-forming characteristics, was sprayed on glass. After letting it dry at room temperature, the foils were floated on the surface of water. The thicknesses, which were determined by weighing several pieces of different size, were found to be on the average 1.5 mg/cm^2 for one set and 1.3 mg/cm^2 for the other set. The suitable combination of these gave the necessary thickness to perform the range measurement.

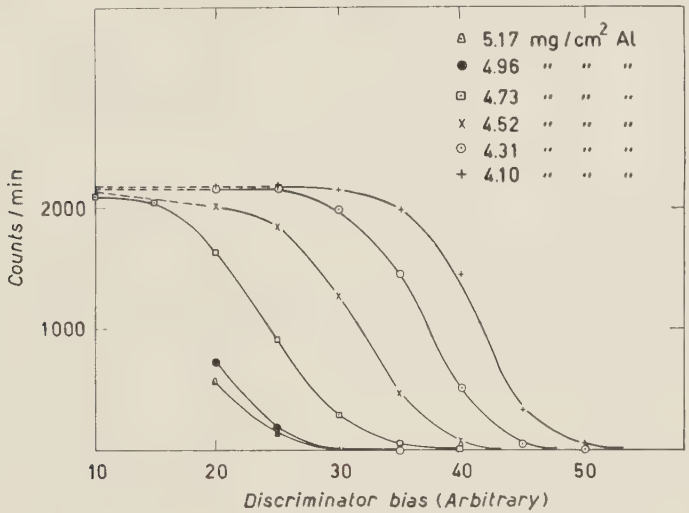

Fig. 3. – Counting rate as a function of bias for different thicknesses of Al absorber.

The range measurements were carried out at a pressure of 20 μm, first with Al to calibrate the system. For this, the range of ^{210}Po α-particles was studied as a function of bias ([3]) (different discriminator levels at the amplifier output to the scaler). To do this, for each thickness of absorber, the counting rate was determined as a function of bias (Fig. 3). Then, from the curves thus obtained, curves of counting rate *vs.* thickness, for a constant bias, were constructed (Fig. 4). Each of these last curves determined an apparent extrapolated range for the corresponding bias. The plot of these apparent ranges against the corresponding values of the bias proved to be linear (Fig. 5) to a rather good approximation, and the extrapolation to zero bias was taken as the range.

([3]) M. G. Holloway and M. S. Livingston: *Phys. Rev.*, **54**, 18 (1938).

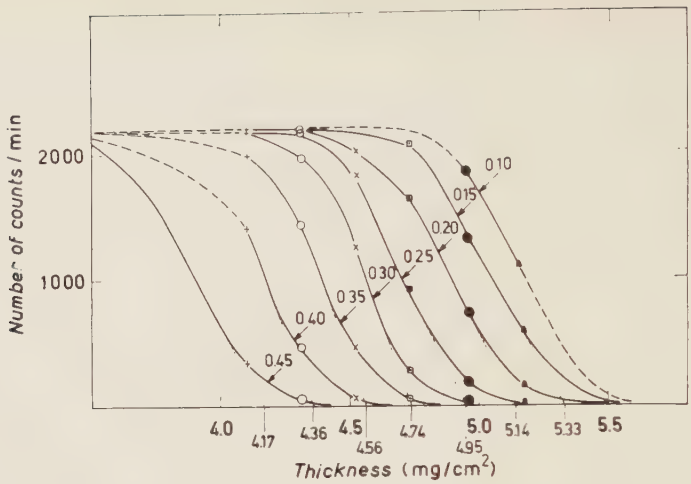

Fig. 4. – Counting rate *vs.* thickness of Al absorber for a given bias with the RCA detector.

The value obtained for Al with this procedure was found to be 5.93 mg/cm², which is slightly less than the expected value for Al (⁴). This was due to a thin layer of Al with which the RCA counter was covered to make it insensitive

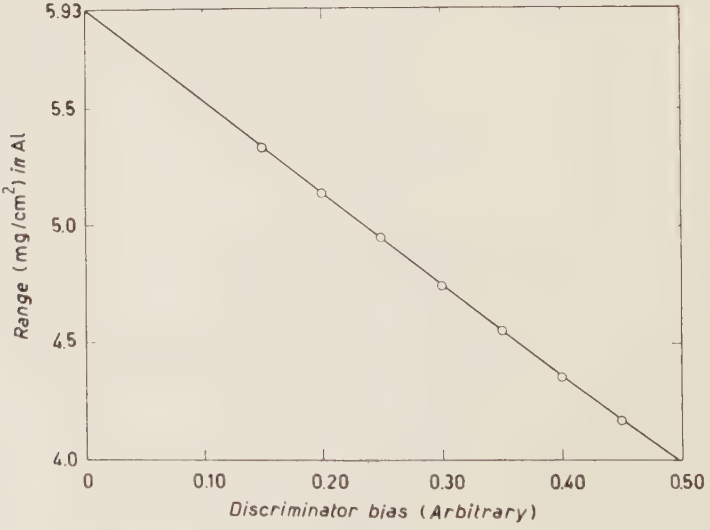

Fig. 5. - Extrapolated ranges in Al for ^{210}Po α-particles as a function of discriminator bias at the scaler. The RCA detector no. 1 was used.

(⁴) I. Joliot-Curie, B. Grinberg, R. Gregoire, B. Pontecorvo and R. J. Walen: *Tables annuelles de constantes et données numeriques*, **26**, Physique Nucléaire (Paris, 1938), p. 44-18 (A4).

to light. The thickness of this coating was determined by using a second semiconductor detector (Hughes No. 179) which was supposed to have no coating. There a similar procedure gave a range in Al of 6.02 mg/cm² for

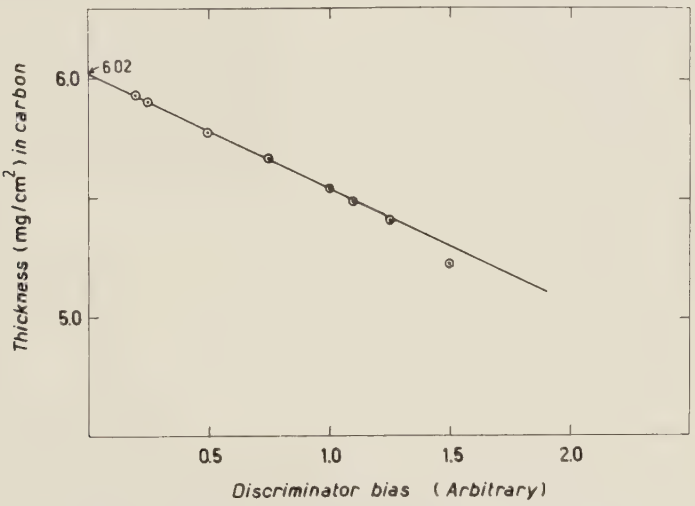

Fig. 6. – Extrapolated ranges in Al for 5.3 MeV ^{210}Po α-particles as a function of discriminator bias at the scaler. The Hughes detector no. 179 was used (compare Fig. 5).

5.3 MeV α-particles (Fig. 6). On this basis, the correction adopted for the thickness of the Al layer on the RCA counter was approximately $\frac{1}{3}$ µm, which is equivalent to 0.09 mg/cm². The value of 6.02 mg/cm² for the range in Al is in reasonable agreement with the value given in reference (4). (The RCA counter was chosen on account of the weak source intensity that was available, since the larger sensitive area of the RCA counter gives it about 20 times the geometrical efficiency of the Hughes counter.)

When the same method was applied to carbon layers, the range in carbon was found to be 4.36 mg/cm². (See Figs. 7 and 8). To this had to be added the carbon equivalent of the 0.09 mg/cm² Al window which was estimated to be 0.066 mg/cm². Thus the value of the range for the ^{210}Po α-particles in C becomes 4.43 mg/cm².

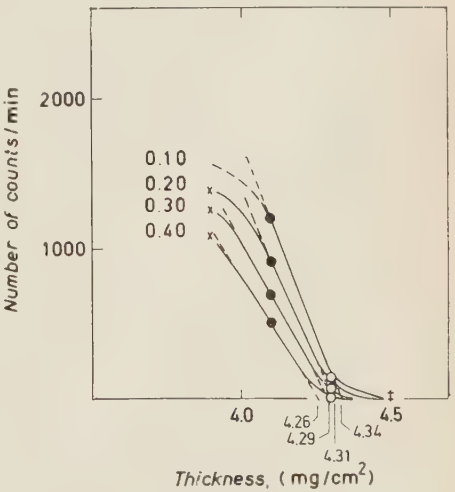

Fig. 7. – Counting rate *vs.* thickness of carbon absorber for a given bias.

By use of the known atomic stopping power of Al relative to air for an α-ray energy of 5.3 MeV ([5]), the value of the atomic stopping power of carbon relative to an air « atom » can be calculated. This was found to be equal to

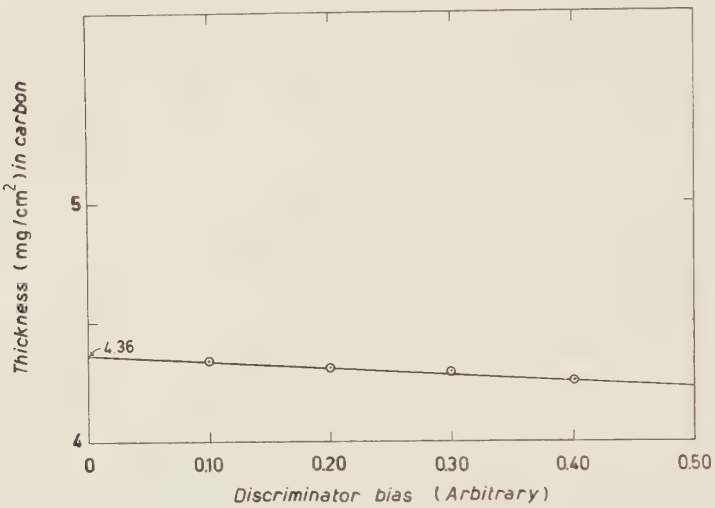

Fig. 8. – Extrapolated ranges in carbon for 5.3 MeV ^{210}Po α-particles as a function of discriminator bias.

0.899 ± 0.009. The error is intended to allow for the uncertainty in correction for the window and for the reproducibility of the measurements. This is in good agreement with values of the relative stopping power found from measurements with compounds by use of Bragg's additive law ([6-12]). The results of YARAMIS ([2]) can perhaps be explained by difficulty with scattering in the foils, the window thickness of his counter, and failure to correct for the dead time in the counter.

([5]) G. MANO: *Ann. Phys., Paris*, **11**, 1, 505 (1934).
([6]) M. CURIE: *Radioactivité*, vol. **1** (Paris, 1935), p. 223.
([7]) G. E. GIBSON and H. EYRING: *Phys. Rev.*, **30**, 553 (1927).
([8]) T. N. HATFIELD, A. E. LOCKENVITZ and M. Y. COLBY: *Journ. Franklin Inst.*, **247**, 133 (1949).
([9]) K. SCHMIEDER: *Ann. Phys.*, **35**, 445 (1939).
([10]) L. H. GRAY: *Proc. Camb. Phil. Soc.*, **40**, 72 (1944).
([11]) M. S. LIVINGSTON and H. A. BETHE: *Rev. Mod. Phys.*, **9**, 245 (1937).
([12]) D. H. WILKINSON: *Proc. Camb. Phil. Soc.*, **44**, 114 (1948).

* * *

The author wishes to express her thanks to Dr. T. H. BRAID for his continuous interest and for providing much of the needed apparatus and sources, to Mr. J. T. HEINRICH for technical assistance and preparing the carbon films, to Dr. M. T. BURGY for guidance and helpful discussions, to Mr. H. M. MANN for the loan of a windowless silicon detector, and to Dr. F. E. THROW for comments on the manuscript.

RIASSUNTO (*)

Abbiamo misurato il percorso delle particelle α del ^{210}Po nel carbonio elementare puro. I fogli di carbonio vennero preparati spruzzando soluzione colloidale di grafite in alcool isopropilico su una lastra di vetro, seccando e facendo poi galleggiare sull'acqua. Come rivelatori si sono usati contatori a semiconduttori. Abbiamo trovato che il percorso è di 4.43 mg/cm^2, che corrisponde ad un valore 0.899+0.009 del potere atomico di arresto del carbonio rispetto all'aria. Questo valore concorda bene con i valori precedenti ottenuti da misure con composti, usando la legge addittiva di Bragg.

(*) *Traduzione a cura della Redazione.*

Branching Ratio of α and β Emissions from ^{212}Bi (ThC) (*).

S. Barkan (**)

Argonne National Laboratory - Argonne, Ill.

(ricevuto il 9 Gennaio 1961)

Summary. — Semiconductor counters, which have good efficiency and energy resolution have been used to measure the branching ratio with increased precision. The measurement was made as usual by comparison of the intensity of the low-energy (< 7 MeV) alphas coming directly from the ^{212}Bi with that of the high-energy (> 7 MeV) alphas coming from the ^{212}Po that is produced by the β-decay of the ^{212}Bi. The average of 8 runs gave $\alpha/(\alpha+\beta) = 0.358 \pm 0.001$.

Measurements of the branching ratio, $\alpha/(\alpha+\beta)$, of ^{212}Bi in the past 25 years have ranged from 33.7% [1] to 36.2% [2]. The most recent values are $(35.9 \pm 0.2)\%$ [3] and $(35.2 \pm 0.3)\%$ [4].

The recent development of semiconductor counters with their good energy resolution and efficiency made it seem quite possible to make this measurement with considerably greater accuracy and accordingly this has been undertaken. The measurement as usual was made by comparison of the intensity of the low-energy (< 7 MeV) alphas coming directly from the ^{212}Bi with that of the high-energy (> 7 MeV) alphas coming from the ^{212}Po, that is produced by the β-decay of the ^{212}Bi.

The experimental set-up used here was the same as the one used to measure the stopping power of carbon [5]. The analyzer and detector arrangement was

(*) Work performed under the auspices of the U. S. Atomic Energy Commission.

(**) On leave from the University of Istanbul.

(1) A. F. Kovarick and N. I. Adams: *Phys. Rev.*, **54**, 420 (1938).

(2) F. E. Sentle, J. A. Farley and N. Lazar: *Phys. Rev.*, **104**, 1629 (1956).

(3) P. Rice-Evans and N. J. Freeman: *Proc. Roy. Soc.* (*London*), A **72**, 303 (1958).

(4) D. Prosperi and S. Sciuti: *Nuovo Cimento*, **9**, 734 (1958).

(5) S. Barkan: *Nuovo Cimento*, **20**, 443 (1961).

calibrated by making use of the ^{239}Pu 5.15 MeV α-ray and the ^{210}Po 5.3 MeV α. On the assumption of a linear response (Fig. 1), it gave energies of 6.00 MeV for ^{212}Bi α-rays and 8.74 MeV for those of ^{212}Po after correction for an energy loss of approximately 40 and 38 keV, respectively, in the 0.09 mg/cm² Al window of the RCA counter. An energy spread of about 1% was observed for the α-rays from ^{212}Po.

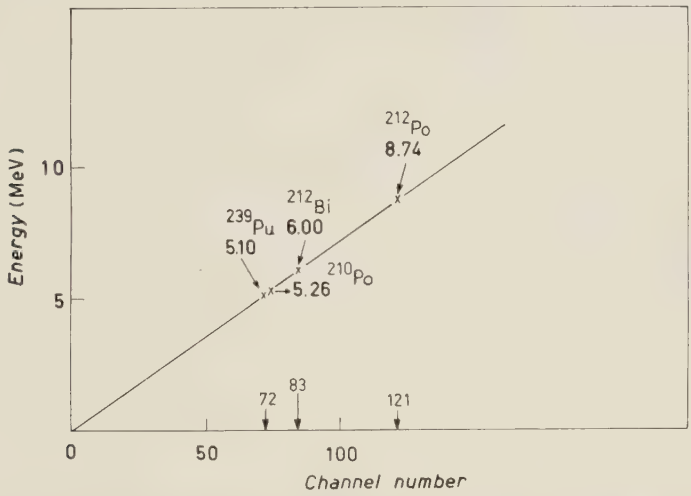

Fig. 1. – Calibration of the 200-channel analyser with the α-rays from ^{239}Pu (5.147 MeV) and ^{210}Po (5.305 MeV). A correction of $\frac{1}{3}\mu \approx 0.09$ mg/cm² was made for the Al window of the RCA detector. This corresponds to an energy loss of approximately 38 keV for ^{212}Po, 40 keV for ^{212}Bi, 45 keV for ^{210}Po, and 47 keV for ^{239}Pu.

The preparation of the source started with ^{228}Th whose decay scheme is as follows:

(RdTh)	(ThX)	(Tn)	(ThA)	(ThB)	(ThC)		(ThC′)
^{228}Th $\xrightarrow{\alpha}$	^{224}Ra $\xrightarrow{\alpha}$	^{220}Rn $\xrightarrow{\alpha}$	^{216}Po $\xrightarrow{\alpha}$	^{212}Pb $\xrightarrow{\beta}$	^{212}Bi	$\xrightarrow{\beta}$	^{212}Po
(1.90 y)	(3.64 d)	(52 s)	(0.16 s)	(10.64 h)	(60.5 min)		(3.0 · 10⁻⁷ s)

^{212}Bi: $\alpha \downarrow$ 6.047 MeV / 6.086 MeV; ^{212}Po: $\alpha \downarrow$ 8.78 MeV

(RaC″)

^{208}Tl $\xrightarrow{\beta}$ ^{208}Pb

(3.1 min) (Stable)

Radiothorium (^{228}Th) was placed in a chamber filled with moist air. Following the decay of the ^{220}Rn gas, the positive ions of the daughter [6] ^{216}Po are

[6] G. H. Briggs: *Phil. Mag.*, **50**, 600 (1925).

driven to the cathode by the electric field between two electrodes about 3 cm apart and maintained at 0 and — 900 V, respectively, in the chamber.

It is possible that some ^{220}Rn atoms would be deposited on the cathode as well. As has been known for a long time ([7,8]), the active deposit of ^{228}Th in a humid atmosphere produces a few particles which form an aerosol and may easily acquire a positive charge and then be adsorbed on the cathode, where the ^{220}Rn decays in two α-emissions to ^{212}Pb. The recoil from an α-emission causes the daughter atom to penetrate into the backing a distance estimated to be of the order of 20 μm in the case of ^{220}Rn on a silver backing ([8,9]). To some extent this can also happen to ^{216}Po which reaches the cathode. This penetration, though very small, certainly tends to reduce the energies of subsequent α-particles which are known to be mainly 6.047 and 6.086 MeV for ^{212}Bi and 8.78 MeV for ^{212}Po ([10]).

The results of the measurements are shown in Fig. 2. In calculating the ratio, 15 channels around each peak were taken. The choice of the channels used on the low-energy side of the peaks is somewhat arbitrary, but the ratio is not very sensitive to this choice.

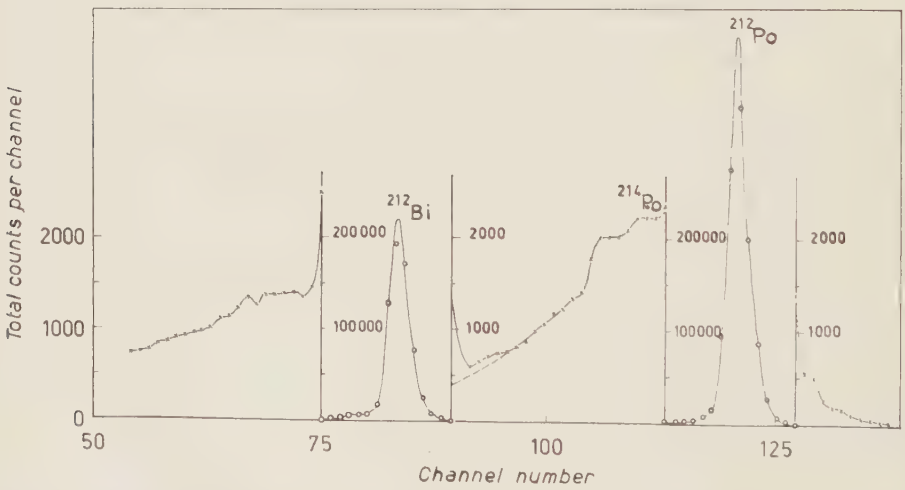

Fig. 2. – α-ray spectrum of ^{212}Bi and ^{212}Po. The ordinate corresponds to the total number of counts per channel for all 8 runs. The extrapolation of the tail of the ^{212}Po α-rays is based on the broken line.

([7]) M. Curie: *Le Radium*, **4**, 381 (1907).
([8]) S. Rosenblum: *Ann. Phys.*, **10**, 408 (1928).
([9]) L. Wertenstein: *Compt. Rend.*, **151**, 469 (1910).
([10]) D. Strominger, J. M. Hollander and G. T. Seaborg: *Rev. Mod. Phys.*, **30**, 585 (1958).

The average of 8 runs gave a value of $\alpha/(\alpha+\beta) = 0.359\,5 \pm 0.000\,8$, where the error is based on the standard deviation of the set of 8 runs. However, a correction for the low-energy tail of ^{212}Po(ThC′) in the ^{212}Bi(ThC) peak must be applied. This tail was extrapolated beneath the ^{212}Bi peak from the counting rate in the 20 channels just above that peak (Fig. 2). This correction amounts to about -0.001 ± 0.001. The corrected value for the ratio is then

$$\frac{\alpha}{\alpha+\beta} = 0.358 \pm 0.001\,.$$

* * *

The author wishes to express her thanks to Dr. T. H. Braid for his continuous interest and advice, to Mr. J. T. Heinrich for technical assistance, to Dr. G. R. Ringo for helpful discussions, and to Dr. F. E. Throw for comments on the manuscript.

RIASSUNTO (*)

Per misurare il rapporto di branching con maggior precisione, abbiamo usato contatori a semiconduttori, che hanno buona efficienza e buona risoluzione energentica. La misura venne eseguita col metodo usuale di confrontare l'intensità delle particelle α di bassa energia (< 7 MeV), provenienti direttamente dal ^{212}Bi, con quella delle particelle α di alta energia (> 7 MeV), provenienti dal ^{212}Po, che si produce nel decadimento β del ^{212}Bi. La media di otto misurazioni diede $\alpha/(\alpha+\beta) = 0.358 \pm 0.001$.

(*) *Traduzione a cura della Redazione.*

Structure of a Quantized Vortex in Boson Systems (*).

E. P. Gross (**)

CERN - Geneva

(ricevuto il 9 Gennaio 1961)

Summary. — For a system of weakly repelling bosons, a theory of the elementary line vortex excitations is developed. The vortex state is characterised by the presence of a finite fraction of the particles in a single particle state of integer angular momentum. The radial dependence of the highly occupied state follows from a self-consistent field equation. The radial function and the associated particle density are essentially constant everywhere except inside a core, where they drop to zero. The core size is the de Broglie wavelength associated with the mean interaction energy per particle. The expectation value of the velocity has the radial dependence of a classical vortex. In this Hartree approximation the vorticity is zero everywhere except on the vortex line. When the description of the state is refined to include the zero point oscillations of the phonon field, the vorticity is spread out over the core. These results confirm in all essentials the intuitive arguments of Onsager and Feynman. The phonons moving perpendicular to the vortex line are coherent excitations of equal and opposite angular momentum relative to the substratum of moving particles that constitute the vortex. The vortex motion resolves the degeneracy of the Bogoljubov phonons with respect to the azimuthal quantum number.

1. – Introduction.

The idea that liquid helium permits macroscopic vortex type motions, as does an ordinary liquid, has played a key role in suggesting and interpreting a large number of recent experiments. The experiments of Vinen [1] provide

(*) Work supported by the Office of Scientific Research, U.S. Air Force and by the National Science Fundation.

(**) Permanent adress, Brandeis University, Waltham, Mass.

(1) H. E. Hall: *Adv. in Phys.*, **9**, 89 (1960); K. R. Atkins: *Liquid Helium* (Cambridge, 1959); W. F. Vinen: *Physica Suppl.*, **24**, 13 (1958).

convincing evidence for the existence of free vortex lines with a circulation quantized in units of h/m. Superposed on a field of vortex motions is the general phonon field. The idea of ONSAGER ([2]) and FEYNMAN ([3]), that circulation is quantized, meets the objection that if vortex excitations of arbitrary circulation and energy were permitted, there would be no superfluidity. On the other hand if there were only phonon-type excitations, it follows from Landau's well known argument based on Galilean invariance, that the critical flow velocity would be much higher than is observed experimentally. With the assumption of quantized vortex lines and rings a qualitative explanation of the low critical velocity becomes possible. The same is true of the behaviour of rotating helium and of many other phenomena.

The description of a vortex quantum mechanically has a number of puzzling aspects. Arguments, for and against the existence of vorticity have been made. None have been precise enough to be universally convincing. Against the existence of vorticity one can argue as follows. In the wave function $\Psi(x_1 \ldots x) = R \exp[i(S/\hbar)]$, S plays the role of a velocity potential when the Schrödinger equation is written in hydrodynamic form. The existence of a potential seems to imply that there can be no vorticity. This has been one objection to attempts to relate macroscopic continuum quantum hydrodynamics with vortex motions to the properties of liquid helium. But it has not been shown, with reasonable definitions of the velocity and vorticity, that the vorticity is everywhere zero. For example, if we define the density as

$$n(\boldsymbol{x}) = \int \Psi^* \Psi \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\, \mathrm{d}\tau \,,$$

the velocity,

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{n(\boldsymbol{x})} \frac{1}{M} \frac{ih}{2} \sum_{i=1}^{N} \int \Psi^* \left\{ \delta(\boldsymbol{x} - \boldsymbol{x}_i) \frac{\hat{c}}{\partial x_i} + \frac{\partial}{\hat{c} x_i} \delta(\boldsymbol{x} - \boldsymbol{x}_i) \right\} \Psi \, \mathrm{d}\tau \,,$$

one would have to show that the vorticity, curl $\boldsymbol{v}(\boldsymbol{x})$, is zero. At best one can make it plausible that if S is slowly varying, and R not very different from the ground state in certain spatial regions, the vorticity is zero. There is an essential distinction between the existence of a potential function $S(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ in $3N$ dimensional configuration space and potential flow of average values of operators in ordinary space.

On the other hand, there is the more convincing reasoning of ONSAGER and FEYNMAN, which indicates that there may be vorticity in concentrated regions. We will try to paraphrase their argument. One may start by con-

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([2]) L. ONSAGER: *Suppl. Nuovo Cimento*, **6**, 2, 249 (1949),

([3]) R. P. FEYNMAN: *Prog. Low Temp. Phys.*, Vol. I (Amsterdam, 1957), ch. 2; *Physica Suppl.*, **24**, 18 (1958).

sidering the flow in a multiply connected region, for example, between concentric cylinders. Corresponding to a given state, say the ground state, there should be other states in which S differs by $\mu \sum_{j=1}^{N} \hbar\vartheta_j$, where μ is an integer. The new motions are those in which each particle has μ units of angular momentum, *i.e.* the system has a total angular momentum $N\hbar\mu$. Without vorticity anywhere, there is nevertheless a circulation. When the inner radius, ϱ, of the region is made small, if it is assumed that the real part of the wave function is still essentially the ground state function, the expectation value of the azimuthal velocity should be proportional to $1/\varrho$, and the vorticity still essentially zero. But the situation becomes unclear as ϱ shrinks to atomic dimensions. An ideal classical line vortex has a vorticity zero everywhere, except on the singular vortex line; and has a characteristic $1/\varrho$ value of the velocity. Quantum mechanically, the $1/\varrho$ behaviour of the velocity can persist as $\varrho \to 0$ only if the real part of the wave function drops essentially to zero on the vortex line. Otherwise the kinetic energy would become infinite. But there must be limitations on the definition of the position of the singular line implied by the uncertainty principle. So, one expects a core in which the density will become small but not necessarily zero, and in which the velocity will be finite. The vorticity should be spread out over the core and should drop continuously and rapidly to zero, as one moves out. That these things happen has not been shown in a detailed theoretical treatment.

Starting from the assumption that there exist such macroscopic excitations with a core of atomic dimensions and with a quantized circulation, but otherwise behaving like a classical vortex, FEYNMAN ([3]) interpreted a large number of phenomena occurring in helium. This point of view was extended and applied with great success by HALL and VINEN ([1]) and others. In spite of the fundamental importance of these physical ideas, both practically and conceptually, very little work has been done to relate the ideas to basic quantum mechanics. Instead attention has been concentrated recently on how the phonon-type excitations, which are relatively well understood as a result of the work of LANDAU, BOGOLJUBOV and FEYNMAN, follow from the many-body Hamiltonian. The purpose of the present paper is to construct a theory of the structure of the simple line vortex, and of the superposed phonon-type excitations when a vortex is present. We shall do this for the case of weakly repelling bosons (or for a dilute gas of hard spheres) in the quantized field description of the many-body problem. In this limit a systematic theory is constructed which follows closely, and is essentially an application of the work of references ([4a]) and ([4b]). It is close in spirit to Bogoljubov's fundamental

([4]) E. P. GROSS: *Ann. Phys.*, *a*) **4**, 57 (1958); *b*) **9**, 292 (1960).

paper ([5]). The results agree in all essentials with the ideas of FEYNMAN and ONSAGER.

The main features are already apparent in the Hartree approximation. That is to say, the possibility of describing a vortex motion in a Bose fluid is contained in this approximation. Each particle is in the same single particle state with angular momentum $\hbar\mu$ about the z-axis. The single particle state is a solution $f^{\mu}(\varrho)\exp[i\mu\vartheta]$ of a self-consistent field (or semiclassical self-interacting field; ref. ([4a])) equation. It has the remarkable property that the density $|f^{\mu}(\varrho)|^2$ is substantially constant at distances greater than a length a. Near the vortex core the density tends to zero like $(\varrho/a)^{2\mu}$. In the semiclassical approximation the expectation value of the azimuthal velocity is strictly proportional to $1/\varrho$, just as for a classical elementary vortex. The core energy per unit length of vortex line is finite. The radius of the core is given very simply as the de Broglie wavelength $a = \hbar/\sqrt{2ME}$, where E is the mean interaction energy per particle $n\int V(x)\,\mathrm{d}^3x$. For a gas of dilute hard spheres, described by a pseudopotential, this is $.15\sqrt{\pi}(\alpha/\sqrt{n\alpha^3})$, where α is the sphere radius and n is the density. These results are obtained in Section **2** from an exact solution of the semiclassical field equation. The connection of such a solution with a Hartree wave function is discussed in ref. ([4a]).

In Section **3** we consider the small oscillations of the semiclassical field about the exact solution, *i.e.* the phonon spectrum in the presence of a vortex. The phonons are described as coherent excitations of pairs of particles of equal and opposite angular momentum relative to the Hartree state of angular momentum $N\hbar\mu$. Far from the cortex core, where the fluid is hardly turning at all, the phonons go over to the usual excitations of Bogoljubov ([5]). The vortex motion removes the degeneracy of the Bogoljubov spectrum with respect to the azimuthal quantum number.

In Section **4** the classical considerations are put in a fully quantum mechanical form. One result is the lowering of the self consistent field estimate of the energy of a vortex line because of the shift in zero point energy of the phonons provided by the normal mode analysis. In addition, consideration of the form of the wave function of a line vortex shows that the zero point motions of the oscillations smear out the density pattern of the Hartree field, yielding a small finite value at the vortex line. At the same time the expectation value of the velocity drops to a finite value (in fact to zero in our approximation), at the vortex line. There is now a definite finite vorticity differing from zero mainly in the core.

The systematic quantum theory has been obtained by first studying in some detail the associated semi classical field theory. Notably, we use special solutions of the semi classical theory to suggest an appropriate single particle

([5]) N. N. BOGOLJUBOV: *Journ. of Phys. USSR*, **9**, 292 (1960).

basis, in terms of which the quantized field is expanded. We use the small oscillation analysis of the classical field to suggest a suitable quasi-particle transformation. This pattern of analysis can be extended to study more general types of vortex excitations. In order to obtain insight into the reason for the success of the hydrodynamic arguments of Feynman, we transcribe the semi-classical theory into hydrodynamic form in Section 5. The main differences from the usual equations of compressible flow are the non-local pressure-density functional relation, and a quantum mechanical pressure term. The latter is responsible for the vortex core structure. Provided the cores are well separated, complicated solutions of the equations can be obtained in the same way as in usual hydrodynamic theory. Each pattern of flow represents a wave packet of macroscopic duration. This opens the way to a consideration of general hydrodynamic flows of a system of bosons, on a basis which has a clear and definite quantum counterpart.

2. – Semi-classical theory [4a].

Consider the boson fluid governed by the Hamiltonian

$$H = \frac{\hbar^2}{2M}\int \nabla\psi^+ \nabla\psi a^3x + \frac{1}{2}\iint \psi^+(\boldsymbol{x})\,\psi^+(\boldsymbol{x}')\,V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\psi(\boldsymbol{x})\,\psi(\boldsymbol{x}')\,\mathrm{d}^3x\,\mathrm{d}^3x' . \tag{2.1}$$

The equation of motion of the field ψ is

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2M}\nabla^2\psi + \psi(\boldsymbol{x})\int V(|\boldsymbol{x}-\boldsymbol{x}'|)/|\psi(\boldsymbol{x}')|^2\,\mathrm{d}^3x' , \tag{2.2}$$

and will be studied as a classical field equation, of the self consistent field type in this section.

Let us look for special exact solutions of the equations of motion possessing cylindrical symmetry. We put

$$\psi(\boldsymbol{x}) = f(\varrho)\exp\left[i\mu\vartheta\right]\exp\left[-i\frac{Et}{\hbar}\right], \tag{2.3}$$

where ϱ, ϑ, z are the cylindrical co-ordinates. The z component of the field angular momentum associated with such a solution is

$$S_z = \frac{\hbar}{2i}\int\left(\psi^+\frac{\partial\psi}{\partial\vartheta} - \frac{\partial\psi^+}{\partial\vartheta}\psi\right)\mathrm{d}^3x = \hbar\mu\int|f|^2(\varrho)\,\mathrm{d}^3x .$$

But the total number of particles is $N = \int\psi^+\psi\,\mathrm{d}^3x = \int|f|^2\,\mathrm{d}^3x$. Thus the total angular momentum is $N\hbar\mu$, *i.e.* $\hbar\mu$ per particle. In addition the local angular

momentum density is constant in time and equal to $|f|^2(\varrho)\hbar\mu$. The component of the velocity of the field at a given space point may be defined as

$$v_\vartheta = \frac{1}{\psi^+\psi}\frac{1}{2M}\frac{\hbar}{i}\left\{\psi^+\frac{1}{\varrho}\frac{\partial\psi}{\partial\vartheta} - \frac{1}{\varrho}\frac{\partial\psi^+}{\partial\vartheta}\psi\right\},$$

and is equal to $\hbar\mu/M\varrho$. It exhibits the characteristic radial dependence of the flow pattern of a classical line vortex. We shall see that in contrast to the usual incompressible (or even compressible) fluid the density $\psi^+\psi = |f(\varrho)|^2$ tends to zero as $\varrho \to 0$ (at the vortex line). However, as $\varrho \to \infty$ the density tends to a constant as is the case for the usual vortex. The vorticity $w_z = (\mathrm{curl}\,\boldsymbol{v})_z = \frac{1}{2}(1/\varrho)(\partial/\partial\varrho)(\varrho V_\vartheta) = 0$ everywhere except at the singular line. But the circulation is $\Gamma = \int v_\vartheta(\mathrm{d}\vartheta\cdot\varrho) = \mu h/M \neq 0$, so that we may write $v_\vartheta = \Gamma/2\pi\varrho$. The radial function $f(\varrho)$ must satisfy the equation

$$Ef = -\frac{\hbar^2}{2M}\left(\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right) - \frac{\mu^2}{\varrho^2}\right)f(\varrho) + f(\varrho)\int V(|\boldsymbol{x}-\boldsymbol{x}'|)^2|f(\varrho')|^2\,\mathrm{d}^3x'\,. \tag{2.4}$$

Since

$$V(|\boldsymbol{x}-\boldsymbol{x}'|) = V\left(\sqrt{(z-z')^2+\varrho^2+\varrho'^2-2\varrho\varrho'\cos(\vartheta-\vartheta')}\right),$$

is invariant to the rotation $\vartheta \to \vartheta+\alpha$, $\vartheta' \to \vartheta'+\alpha$, the assumed separation of variables is indeed consistent.

To examine the behaviour of the function $f(\varrho)$, we consider first for simplicity the case of a short range potential, in fact $V(|\boldsymbol{x}-\boldsymbol{x}'|) = V\,\delta(\boldsymbol{x}-\boldsymbol{x}')$. We have in mind, more precisely, a pseudopotential ([6]). At this stage of the theory it can be treated as a δ-function. Then

$$Ef = -\frac{\hbar^2}{2M}\left(\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right) - \frac{\mu^2}{\varrho^2}\right)f + Vf|f(\varrho)|^2\,, \tag{2.5}$$

with

$$\int |f(\varrho)|^2\,\mathrm{d}^3x = N\,.$$

(For the s-wave pseudopotential $V = 8\pi\alpha(\hbar^2/M)$, where α is the radius of the hard spheres.)

$f(\varrho)$ may be taken as real. For small ϱ, the centrifugal force term dominates and

$$f(\varrho) \to j_\mu\left(\sqrt{\frac{2ME}{\hbar^2}}\,\varrho\right) \qquad \text{as} \qquad \varrho \to 0\,. \tag{2.6}$$

([6]) K. Huang and C. N. Yang: *Phys. Rev.*, **105**, 767 (1957); K. Huang, T. D. Lee and C. N. Yang: *Phys. Rev.*, **106**, 1136 (1975).

As $\varrho \to \infty$

$$f(\varrho) \to f_0 = \text{constant} . \tag{2.7}$$

We must have $E = Vf_0^2$. The last remaining constant, f_0^2, is fixed by the normalization condition $\int |f^2| \mathrm{d}^3x = N$. This has a small finite contribution from the core of the vortex as $N \to \infty$, $\Omega \to \infty$. In this limit

$$f_0^2 \to \frac{N}{\Omega} = \frac{N}{L\pi R^2} ,$$

where L is the extent of the system in the z direction and R is the radius of the cylinder of « quantization ». Thus f_0^2 is the mean number density. The behaviour of f as $\varrho \to \infty$ is quite remarkable. It is brought about by the non linear term $Vf|f|^2$, *i.e.* the self-consistent field. It is physically clear that the repulsive forces should force the system to uniform density almost everywhere.

The energy of the fluid is obtained by substituting the solution $f^{\mu}(\varrho)$ in H

$$H = -\frac{\hbar^2}{2M}\int f^{\mu}(\varrho)\left\{\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right) - \frac{\mu^2}{\varrho^2}\right\} f^{\mu}(\varrho)\,\mathrm{d}^3x + \tag{2.8}$$

$$+ \frac{1}{2}\int\int (f^{\mu}(\varrho))^2\, V(\boldsymbol{x}-\boldsymbol{x}')\,(f^{\mu}(\varrho'))^2\,\mathrm{d}^3x\,\mathrm{d}^3x' .$$

For the vortex-free case ($\mu = 0$) the lowest state is $f^0 = f_0 = \text{const.}$, with an energy $H = \frac{1}{2} f_0^4 \int V(s)\,\mathrm{d}^3s$. When a vortex is present ($\mu^0 = 0$), $f^{\mu}(\varrho) = f_0$ everywhere and there is a finite correction to the potential energy per unit length of the vortex, arising from the core. The behaviour of the kinetic energy is, however, more important. If the kinetic energy is evaluated with $f^{\mu}(\varrho) = f_0$ everywhere, the result is divergent, as $\varrho \to 0$. However, the actual $f^{\mu}(\varrho) \to 0$, as $\varrho \to 0$, so that the density of fluid tends to zero as the velocity tends to ∞, in such a way that the kinetic energy per unit length of the vortex core is finite. The region outside the core ($\varrho > a$) contributes a kinetic energy

$$\frac{\hbar^2}{2M}\mu^2 \int_a^R \frac{f_0^2}{\varrho^2}\,\varrho\,\mathrm{d}\varrho \cdot 2\pi L = \frac{\hbar^2\mu^2}{2M} f_0^2 \cdot 2\pi L \ln\frac{R}{a} = \frac{MI^2}{8\pi^2} f_0^2 \ln\frac{R}{a}\,(2\pi L) ,$$

i.e., the characteristic logarithmic dependence on the outer radius of the vortex.

For a more general, but still essentially short-range potential, $f(\varrho)$ still tends to a constant f_0 with

$$E = f_0^2 \int V(S)\,\mathrm{d}^3S . \tag{2.9}$$

The behaviour for $\varrho \to 0$ is again the same as for a δ-function potential, and is determined entirely by the centrifugal potential.

It is of course not easy to find exact solutions of the self consistent field equation for $f^{\mu}(\varrho)$. However, it is easy to find an accurate estimate of the size of the vortex core. We use our knowledge of the exact behaviour at small and large distances and assume that

$$f^{\mu}(\varrho) = A j_{\mu}\left(\sqrt{\frac{2ME}{\hbar^2}}\, a\right), \qquad \varrho < a, \tag{2.10}$$

$$= f_0, \qquad \varrho > a.$$

A is fixed by the continuity requirement at $\varrho = a$,

$$A = f_0 / j_{\mu}\left(\sqrt{\frac{2ME}{\hbar^2}}\, a\right). \tag{2.11}$$

The continuity of the radial derivatives of f requires

$$\frac{\mathrm{d}j_{\mu}}{\mathrm{d}\varrho}\left(\sqrt{\frac{2ME}{\hbar^2}}\, \varrho\right)/\varrho = a = 0. \tag{2.12}$$

This fixes the core size a_{μ} in terms of the position of the first zero of $\mathrm{d}j^{\mu}/\mathrm{d}\varrho$. We have

$$\left.\begin{array}{lll} \mu = 1 & a_1 = \dfrac{\hbar}{\sqrt{2ME}} & \\ \mu = 2 & a_2 = & \pi(.59), \\ \mu \text{ large} & a_{\mu} = & \pi(.97), \\ & & (\mu + .26\pi\mu^{\frac{1}{3}}). \end{array}\right.$$

For $\mu = 1$ the core size is of the order of the de Broglie wavelength associated with the mean energy of interaction per particle $E = V f_0^2$. For a dilute gas of hard spheres of radius α, mass M, $V = 8\pi(\alpha\hbar^2/M)$ and we have

$$a' = \left(\frac{.59}{4}\sqrt{\pi}\right)\frac{\alpha}{\sqrt{\alpha^3 f_0^2}},$$

i.e.: the vortex radius is larger then the hard sphere radius by the factor

$$\frac{1}{4}\left(\frac{\text{particle separation}}{\text{sphere radius}}\right)^{\frac{3}{2}}.$$

This indicates that in a more dense gas, the vortex would have a size about equal to the interparticle spacing. But we cannot be sure for the zero point motions of the phonons are as important as the self-consistent field.

The energy associated with this approximate solution may be obtained readily. We define the pure numbers

$$\text{(2.13)}\qquad \begin{cases} C_\mu = \dfrac{\int_0^{S_\mu} j_\mu^2 S\,\mathrm{d}S}{j_\mu^2(S_\mu)}\,, \qquad D_\mu = \dfrac{\int_0^{S_\mu} j_\mu^4 S\,\mathrm{d}S}{j_\mu^4(S_\mu)}\,, \\ C_\mu = \dfrac{S_\mu^2}{2}\left[1 - \dfrac{j_0(S_\mu)\, j_1(S_\mu)}{j_1^2(S_\mu)}\right], \end{cases}$$

where S^μ are the roots $S_1 = (.59)$, $S_2 = (.97)$, ... Then the normalization of $\int f^2\,\mathrm{d}^3x = N$ yields

$$\text{(2.14)}\qquad \frac{N}{2\pi L} = f_0^2 \left\{ \frac{a^2}{S_\mu^2} C_\mu + \frac{R^2 - a^2}{2} \right\}.$$

The energy is

$$\text{(2.15)}\qquad \frac{H}{2\pi L} = f_0^2 \frac{\hbar^2}{2M}\left(C_\mu + \frac{D_\mu}{2}\right) + \frac{V f_0^4}{2}(R^2 - a^2) + \frac{\hbar^2}{2M}\ln\frac{R}{a}\,.$$

In addition to the potential energy of the quiescent fluid and the logarithmic vortex energy there is the finite correction (as $R \to \infty$) per unit length of the vortex core. If we write $\ln(R/a) = \ln(R/b) - \ln(a/b)$, we can make the corrections vanish by taking

$$C_\mu + \frac{D_\mu}{2} - \frac{1}{4} = \ln\frac{a}{b}\,; \qquad b_1 \sim \frac{a_1}{5}\,.$$

Then b is a « pseudo » radius of the vortex core.

We might improve the above estimates by taking a more refined trial function $f^\mu(\varrho)$ and using the variation principle for the energy with f subject to $\int f^2\,\mathrm{d}^3x = N$. But the results differ from the above in only unimportant ways. It is more important to consider the possibility, that for a given μ, there exist other solutions, with nodes, of the self-consistent field equations. This requires more detailed study. If they exist, they would have to be stable to be directly relevant.

We note in conclusion that the results of this section may be obtained directly in configuration space. The Hartree approximation is $\Psi^\mu(\boldsymbol{x}, ..., \boldsymbol{x}_N) = (1/\sqrt{N}) \prod_{j=1}^{N} f^\mu(\varrho_j) \exp[i\mu\vartheta_j]$, and represents a fixed number of particles. The function $f^\mu(\varrho_j)$ is the same one that is used in the field picture. The advantage of the field point of view appears in the introduction of quasi-particles, as discussed in the next section. This is done at the expense of working with an indefinite number of particles, and an indefinite angular momentum.

3. – Small oscillations of the fluid about a vortex motion.

We have shown that there is an exact solution of the classical equations of motion involving circulation. It differs from a hydrodynamic vortex in that the structure of the core is fully determined, and the density goes to zero at the singular vortex line.

In the classical theory of the wave field, the next step is to examine the small oscillations of the system about the exact solution. We put

$$\psi = \exp\left[-i\frac{Et}{\hbar}\right]\{f^{\mu}(\varrho)\exp[i\mu\vartheta] + \varphi(\boldsymbol{x}, t)\}\,, \tag{3.1}$$

and linearize the equations of motion. One finds for a δ-function potential

$$i\hbar\dot{\varphi} = -\frac{\hbar^2}{2M}\nabla^2\varphi + (V|f(\varrho)|^2 - E)\varphi + Vf^2(\varrho)\exp[i\mu\vartheta]\cdot \cdot\{\exp[-i\mu\vartheta]\varphi + \exp[i\mu\vartheta]\varphi^+\}\,, \tag{3.2}$$

together with the complex conjugate equation.

It corresponds to the Hamiltonian

$$H_2 = -\frac{\hbar^2}{2M}\int\varphi^+\nabla^2\varphi\,\mathrm{d}^3x + V\int f^2(\varrho)\{\varphi^+\exp[i\mu\vartheta] + \varphi\exp[-i\mu\vartheta]\}^2\,\mathrm{d}^3x\,, \tag{3.3}$$

which is positive definite. The same result holds for more general positive interactions $V(|\boldsymbol{x}-\boldsymbol{y}|)$.

To analyse the normal mode spectrum, we define a complete set of orthonormal functions as the eigenfunctions of the linear operator

$$L^{\mu} = -\frac{\hbar^2}{2M}\nabla^2 + V|f^{\mu}(\varrho)|^2 - E\,. \tag{3.4}$$

We shall continue to work with the δ-function, in spite of the fact that one must use a pseudopotential to avoid a divergent total zero-point energy from the shifted normal modes. The spectrum itself is quite definite and finite.

We write

$$L^{\mu}\varphi^{\mu}_{\varkappa,\sigma,m} = \left(E^{\mu}_{\sigma,m} - E + \frac{\hbar^2\varkappa^2}{2M}\right)\varphi^{\mu}_{\varkappa,\sigma,m}\,, \tag{3.5}$$

where

$$\text{(3.6)}\qquad \begin{cases} \varphi_{\varkappa,\sigma,m} = \dfrac{1}{\sqrt{2\pi L}} \exp\left[i\varkappa z\right] \exp\left[im\vartheta\right] g^{\mu}_{\sigma,m}(\varrho)\,, \\ \int g^{*\mu}_{\sigma,m}(\varrho)\, g^{\mu}_{\sigma',m}(\varrho)\,\varrho\,\mathrm{d}\varrho = \delta_{\sigma,\sigma'} \end{cases}$$

The superscript μ is to emphasize that the set of functions depends on the state of vortex excitation. The eigenvalues E^{μ}_{σ} depend on $|m|$ because of the cylindrical symmetry of the potential $V|f^{\mu}(\varrho)|^2$. We will work with standing waves. The functions $g^{\mu}_{\sigma,m}(\varrho)$ can be taken to be real and defined in a large cylindrical region, with the boundary conditions

$$\text{(3.7)}\qquad g^{\mu}_{\sigma,m}(\varrho = R) = 0\,.$$

We note that

$$g^{\mu}_{0,\mu} = \frac{1}{\sqrt{N}} f^{\mu}(\varrho)\,, \qquad E^{\mu}_{0,\mu} = E\,,$$

where we write $\sigma = 0$ for the lowest possible value of σ. We also have

$$g^{\mu}_{\sigma,m} = g^{\mu}_{\sigma,-m}\,; \qquad \varphi^{*\mu}_{\varkappa,\sigma,m} = \varphi^{\mu}_{-\varkappa,\sigma,-m}\,.$$

Let us now expand

$$\text{(3.8)}\qquad \begin{cases} \varphi = \sum a_{\varkappa,\sigma,m}\varphi_{\varkappa,\sigma,m}\,, \\ \varphi^{+} = \sum a^{+}_{\varkappa,\sigma,m}\varphi^{*}_{\varkappa,\sigma,m} = \sum a^{+}_{-\varkappa,\sigma-m}\varphi_{\varkappa,\sigma,m}\,. \end{cases}$$

Then eq. (3.2) is equivalent to

$$\text{(3.9)}\qquad \left\{i\hbar\frac{\partial}{\partial t} - \left(E^{\mu}_{\sigma,m} - E + \frac{\hbar^2\varkappa^2}{2M}\right)\right\} a_{\varkappa,\sigma,m} = \\ = \sum_{\sigma'} \left\{F^{\mu}(\sigma m\,|\,\sigma' m) a_{\varkappa,\sigma',m} + F\ (\sigma m\,|\,\sigma', 2\mu - m) a^{+}_{-\varkappa,\sigma',2\mu-m}\right\},$$

where

$$F^{\mu}(\sigma m\,|\,\sigma' m') = V\int g^{\mu}_{\sigma,m} g^{\mu}_{\sigma',m'} (f^{\mu})^2\,\varrho\,\mathrm{d}\varrho\,, \qquad F^{*} = F\,.$$

It follows that

$$\text{(3.10)}\qquad \left\{-i\hbar\frac{\partial}{\partial t} - \left(E^{\mu}_{\sigma,2\mu-m} - E + \frac{\hbar^2\varkappa^2}{2M}\right)\right\} a^{+}_{-\varkappa,\sigma,2\mu-m} = \\ = \sum_{\sigma'} \left\{F^{\mu}(\sigma, 2\mu - m\,|\,\sigma', 2\mu - m) a^{+}_{-\varkappa,\sigma',2\mu-m} + F^{\mu}(\sigma, 2\mu - m\,|\,\sigma', m) a_{\varkappa,\sigma',m}\right\}.$$

The dominant feature of these equations is the strong coupling of $a_{\varkappa,\sigma,m}$ and $a^{+}_{-\varkappa,\sigma,2\mu-m}$. There is a less important coupling of the radial modes. In the absence of vortex motion ($\mu = 0$), the coupling is that of equal and opposite linear momenta, *i.e.* Bogoljubov's theory, here described in cylindrical co-ordinates. When there is a vortex, the annihilation operator of angular momentum $m - \mu$ relative to the vortex angular momentum is strongly coupled to the creation operator of angular momentum $-(m-\mu)$ relative to the vortex. This actually involves a net loss of angular momentum in the small amplitude approximation. In the quantum version the state vector for the vortex contains pairs of particles of equal and opposite angular momentum excited out of the vortex « reservoir ». There is a loss of 2μ units of angular momentum for each pair, since it comes from the moving substratum which is treated as fixed. The state is one of indefinite angular momentum and is analogous to, but in addition to, the description in terms of an indefinite number of particles in the usual boson theory.

Let us consider only the terms in which $\sigma' = \sigma$, thus neglecting the radial mode coupling. We look for solutions where a and a^{+} have the time behaviour $\exp[-i\varepsilon t/\hbar]$. (The fact that this violates the requirement that φ^{+} be the hermitian conjugate of φ is taken care of by the fact that there is a corresponding solution $\exp[+i\varepsilon t/\hbar]$.)

$$(3.11) \qquad a_{\varkappa,\sigma,m} = A_{\varkappa,\sigma,m}\exp[-i\varepsilon t/\hbar]\,, \qquad a^{+}_{\varkappa,\sigma,m} = \exp[-i\varepsilon t/\hbar]A^{+}_{\varkappa,\sigma,m}\,.$$

The energy ε is given as

$$(3.12) \qquad \begin{aligned} &+ \Delta_{\sigma,\varkappa,2\mu-m} - \Delta_{\sigma,\varkappa,m}\,, \\ \varepsilon^2 + \varepsilon\{F(\sigma,\,2\mu-m) - F(\sigma m\,|\,\sigma m)\} &= \\ &= \{\Delta_{\sigma,\varkappa,2\mu-m} + F(\sigma,\,2\mu-m\,|\,\sigma,\,2\mu-m)\}\{\Delta_{\sigma,\varkappa,m} + F(\sigma m\,|\,\sigma m)\} - \\ &- F(\sigma m\,|\,\sigma,\,2\mu-m)\,F(\sigma,\,2\mu-m\,|\,\sigma,\,m)\,, \end{aligned}$$

$$\Delta_{\sigma,\varkappa,m} = E^{\mu}_{\sigma,m} - E + \frac{\hbar^2\varkappa^2}{2M}\,.$$

The eigenvector is

$$(3.13) \qquad \frac{A_{\varkappa,\sigma,m}}{A^{+}_{-\varkappa,\sigma,2\mu-m}} = \frac{F(\sigma m\,|\,\sigma,\,2\mu-m)}{\varepsilon - \Delta_{\sigma,\varkappa,m} - F(\sigma m\,|\,\sigma m)}\,.$$

In complete analogy with the usual boson theory, the variables

$$(3.14) \qquad \begin{cases} b_{\varkappa,\sigma,m} = \cosh\gamma\, a_{\varkappa,\sigma,m} + \sinh\gamma\, a^{+}_{-\varkappa,\sigma,2\mu-m}\,, \\ b^{+}_{-\varkappa,\sigma,2\mu-m} = \sinh\gamma\, a_{\varkappa,\sigma,m} + \cosh\gamma\, a^{+}_{-\varkappa,\sigma,2\mu-m}\,, \\ \gamma = \gamma_{\varkappa,\sigma,m}\,, \end{cases}$$

are connected by a canonical transform to the a, a^+. They oscillate harmonically as

$$i\hbar\,\dot{b}_{\varkappa,\sigma,m} = \varepsilon_{\varkappa\sigma m}\, b_{\varkappa,\sigma,m}\,.$$

(Compare eq. (4.12) to (4.16) for the values of $\gamma_{\varkappa,\sigma,m}$.)

For the case that there is no vortex, we should recover the results of Bogoljubov. Let us review how this comes about. Our treatment is unfamiliar, first because we are using cylindrical co-ordinates, and second because we have expanded in standing waves. The first step has been the determination of the self-consistent field $f^\mu(\varrho)$. The function $f^0(\varrho)$ obeys equation (2.5) with $\mu = 0$. It is independent of ϱ as $\varrho \to 0$. We have also $E = V(f^0)^2$. But with the boundary condition (3.7), as $\varrho \to R$, f^0 must adjust to our demand $f^0(\varrho = R) = 0$. The function drops to zero within a distance $\hbar/\sqrt{2ME}$ of the cylinder of quantization. This is the same characteristic length that occurs in the core size argument. Thus the ground state contains a finite fraction of the particles in a single-particle state which is essentially a constant except near the boundary. This is substantially the starting point of the usual theory as developed with periodic boundary conditions. The details of $f^0(\varrho)$ near the boundary complicate the analysis, but are not essential for many purposes. It is clear that the semiclassical starting point offers a natural way of treating general boundary conditions. The highly occupied single-particle state, as determined by a self-consistent field equation, tends to uniform density except near the boundary. One can impose the condition that the wave function vanish, just as one can impose periodic conditions, without creating artificial difficulties which occur in other methods. These difficulties arise because the standing-wave solutions of the Schrödinger equation for non-interacting particles have large density variations throughout the box.

When there is no vortex, the functions $g^0_{\sigma,m}(\varrho)$ are ordinary Bessel functions $j_m(\sigma\varrho)$, with the quasi-continuous values of σ determined by $j_m(\sigma R) = 0$ and with $E_\sigma - E = \hbar^2\sigma^2/2M$. The energy normal-mode-frequencies are

$$\varepsilon^2 = \frac{\hbar^2}{2M}(\sigma^2 + \varkappa^2)\left\{\frac{\hbar^2}{2M}(\sigma^2 + \varkappa^2) + 2F^0(\sigma m\,|\,\sigma m)\right\}. \tag{3.15}$$

Now the matrix element

$$F^0(\sigma m\,|\,\sigma m) = \int g^0_{\sigma m}(\varrho)\, g^0_{\sigma' m}(f^0)^2 \varrho\, \mathrm{d}\varrho\,,$$

is just $\delta_{\sigma,\sigma'}(f^0)^2$ and is independent of m. Thus the frequency spectrum of the excitations is degenerate with respect to the azimuthal quantum number. Eq. (3.15) is of course Bogoljubov's spectrum, as written in cylindrical co-ordinates. In the special case of no vortex, the entire Hamiltonian H_2 is made

diagonal by the simple normal mode transformation, *i.e.* there is no radial mode coupling. Thus we see how the usual theory is contained in the present description.

When there is a quantized vortex, there are modifications of the normal mode frequencies and eigenvectors. The most striking feature of eq. (3.12) is the removal of the m degeneracy. The only remaining degeneracy is that eq. (3.12) is invariant to the substitution $\varepsilon \to -\varepsilon$, $m \rightleftarrows 2\mu - m$. This reflects the possibility of forming standing waves relative to the vortex. The modes with different m values represent different motions relative to the vortex and are therefore perturbed differently.

A more detailed study of the normal mode spectrum may be made if one takes a crude approximation to the « self-consistent » potential

$$U(\boldsymbol{x}) = \int V(|x - x'|)\,|f^{\mu}(\varrho')|^2\,\mathrm{d}^3x' ,$$

$$\text{(3.16)} \qquad \begin{cases} U(\boldsymbol{x}) \sim 0 & \text{for } \varrho < a , \\ E & \text{for } \varrho > a , \end{cases}$$

where a depends on μ.

Thus a cylindrical well will be used to generate an approximation to the orthonormal basis $\varphi_{\kappa,\sigma,m}$. This leads to

$$\text{(3.17)} \qquad \begin{cases} g^{\mu}_{\sigma,m} = \alpha j_m(\sigma' a) , & \varrho < a , \\ \quad = \beta j_m(\sigma\varrho) + \gamma n_m(\sigma\varrho) , & \varrho > a , \end{cases}$$

where

$$\sigma' = \left(\frac{2ME_{\sigma}}{\hbar^2}\right)^{\frac{1}{2}}, \qquad \sigma = \left(\frac{2M}{\hbar^2}\,(E_{\sigma} - E)\right)^{\frac{1}{2}},$$

and α, β, γ depend on σ and m. The conditions that $g_{\sigma,m}$ and its radial derivative be continuous at $\varrho = a$ fix β and γ in terms of α, *i.e.*

$$\text{(3.18)} \qquad \begin{cases} \alpha j_m(\sigma' a) = \beta j_m(\sigma a) + \gamma n_m(\sigma a) , \\ \alpha\sigma' \dfrac{\mathrm{d}j_m}{\mathrm{d}(\sigma' a)} = \beta\sigma \dfrac{\mathrm{d}j_m(\sigma a)}{\mathrm{d}(\sigma a)} + \gamma\sigma \dfrac{\mathrm{d}n_m(\sigma a)}{\mathrm{d}(\sigma a)} . \end{cases}$$

In turn, α is fixed by the requirement that $g_{\sigma,m}$ be normalized. The possible values of σ are determined by

$$\text{(3.19)} \qquad g_{\sigma,m}(R) = \beta j_m(\sigma R) + \gamma n_m(\eta R) = 0 ,$$

There are of course shifts from the vortex free values in the permitted values of σ of order $1/R$, and associated shifts in $E_\sigma = E + (\hbar^2\sigma^2/2M)$. There will be additional shifts in the normal mode frequencies $\varepsilon_{\varkappa,\sigma,m}$. This gives rise to a change in zero point energy. We can compute the difference in zero-point energies with and without a vortex as

$$\frac{1}{2}\sum_{\varkappa\sigma m}(\varepsilon^{\mu}_{\varkappa,\sigma,m} - \varepsilon^{0}_{\varkappa,\sigma}), \tag{3.20}$$

using eq. (3.12).

The examination of the phase-shifts is closely related to the theory of the scattering of a phonon from a vortex. It is sufficiently interesting to warrant separate treatment, which will be carried out in a separate paper.

Let us now sketch the modifications to be expected for a more general potential. The operator L^μ becomes

$$L^\mu = -\frac{\hbar^2}{2M}\nabla^2 + \int V(|\boldsymbol{x}-\boldsymbol{x}'|)\,|f^\mu(\varrho')|^2\,\mathrm{d}^3x' - E\,. \tag{3.21}$$

The « potential » is again cylindrically symmetrical. We expand

$$V(|\boldsymbol{x}-\boldsymbol{x}'|) = \frac{1}{2\pi L}\sum_{\varkappa',m'} W_{\varkappa',m'}(\varrho,\varrho')\exp[i\varkappa(z-z')]\exp[im'(\vartheta-\vartheta')]. \tag{3.22}$$

The « potential » is

$$\int W_{0,0}(\varrho,\varrho')\,|f^\mu(\varrho')|^2\varrho'\,\mathrm{d}\varrho'\,.$$

The equation for the small oscillations is

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2M}\nabla^2\right)\varphi = \left\{\int V(\boldsymbol{x}-\boldsymbol{x}')\,|f^\mu(\varrho')|^2\,\mathrm{d}^3x' - E\right\}\varphi + \tag{3.23}$$

$$+ f^\mu(\varrho)\exp[i\mu\vartheta]\int V(x-x')f^\mu(\varrho')\{\exp[-i\mu\vartheta']\,q(x') + \exp[i\mu\vartheta']\,q^+(x')\}\,\mathrm{d}^3x'\,.$$

Expanding in terms of the basis $\varphi_{\varkappa,\sigma,m}$, we find eq. (3.9) and (3.10) with the replacements

$$F^\mu(\sigma m\,|\,\sigma' m') \to \mathscr{F}^\mu_\varkappa(\sigma m\,|\,\sigma' m')\,,$$

$$\mathscr{F}^\mu_\varkappa = \iint \varrho\,\mathrm{d}\varrho\,\varrho'\,\mathrm{d}\varrho' f^\mu(\varrho)\,g^\mu_{\sigma,m}(\varrho)\,W_{\varkappa,m-\mu}(\varrho,\varrho')f^\mu(\varrho')\,g_{\sigma',m'}(\varrho')\,. \tag{3.24}$$

The rest of the theory is then developed as before.

4. – Quantum theory.

There are many, essentially related ways, of developing the quantum theory of weakly interacting bosons which yield identical results in the lowest approximations. We adopt the general approach of Section **4** of reference ([1b]), which is in particularly close correspondence to the semiclassical theory. In this approach the main new point in the quantum theory is that the approximate eigenstates no longer correspond to a sharp value for certain constants of the motion. Thus the classical solution $\exp[-iEt/\hbar]f^{\mu}(\varrho)\exp[i\mu\vartheta]$ corresponds to fixed, time-independent values of the number functional $N_{\text{op}}=\int\psi^{+}\psi\,\mathrm{d}^3x$ and the angular momentum functional $J_z=(\hbar/2i)\int(\psi^{+}(\partial\psi/\partial\vartheta)-(\partial\psi^{+}/\partial\vartheta)\psi)\,\mathrm{d}^3x$. In the quantum theory, the approximate eigenstates of H have given expectation values for the operators N_{op} and J_z, but are not exact eigenstates. This is particularly apparent in the case of the number operators. The phase factor $\exp[-iEt/\hbar]$ may be removed from the field operators by the time-dependent canonical transformation $U=\exp[i(Et/\hbar)N_{\text{op}}]$ which changes the Hamiltonian to $H-EN_{\text{op}}$. This is equivalent, in the quantum theory to introducing the Lagrange multiplier E and determining it by the condition that the expectation value of N_{op} is N. To find approximate eigenstates of the Hamiltonian which are also exact eigenstates of N and J_z, one has to apply projection operators in the present formalism. For the vortex state, they are of the order of N particles in a single-particle state of angular momentum $\hbar\mu$ with a vortex-type radial dependence. This amounts to a total angular momentum of $N\hbar\mu$. In the quantum theory we ought to introduce another Lagrange multiplier ω, to be determined by the requirement that the expectation value of J_z is $N\hbar\mu$

We shall therefore study the effective Hamiltonian $\mathscr{H}=H-EN_{\text{op}}-\omega(J_z)_{\text{op}}$. The quantized field is expanded in a complete basis similar to that of Section **3**. The Hamiltonian is

$$\mathscr{H}=\sum\{(K\sigma m\,|\,T\,|\,K\sigma' m)-(E+\hbar\omega m)\,\delta_{\sigma,\sigma'}\}\,a^{+}_{K\sigma m}a_{K\sigma' m}+ \tag{4.1}$$
$$+\sum(K\sigma m;\,K'\sigma' m'\,|\,G\,|\,K''\sigma'' m'';\,K'''\sigma''' m''')\,a^{+}_{K\sigma m}a^{+}_{K'\sigma' m'}a_{K''\sigma'' m''}a_{K'''\sigma''' m'''}\,.$$

Here

$$(K\sigma m\,|\,T\,|\,K'\sigma' m')\equiv-\frac{\hbar^2}{2M}\int\varphi^{*}_{K\sigma m}\nabla^2\varphi_{K'\sigma' m'}\,\mathrm{d}^3x= \tag{4.2}$$
$$=\delta_{m,m'}\delta_{K,K'}\left(-\frac{\hbar^2}{2M}\right)\int g^{*}_{\sigma,m}(\varrho)\left\{D_\varrho-\frac{m^2}{\varrho^2}-K^2\right\}g_{\sigma' m}\,\varrho\,\mathrm{d}\varrho\,,$$
$$D_\varrho=\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right),$$

and

$$\langle K\sigma m;\, K'\sigma' m' \,|\, G \,|\, K''\sigma'' m'';\, K'''\sigma''' m''' \rangle = \tag{4.3}$$

$$= \frac{1}{2}\iint V(x-y)\varphi^*_{K\sigma m}(x)\varphi^*_{K'\sigma' m'}(y)\varphi_{K''\sigma'' m''}(x)\varphi_{K'''\sigma''' m'''}(y)\,\mathrm{d}^3x\,\mathrm{d}^3y =$$

$$= \frac{1}{2}\frac{1}{2\pi L}\sum\iint \varrho\,\mathrm{d}\varrho\,\varrho'\,\mathrm{d}\varrho'\,W_{L,n}(\varrho,\varrho')g_{\sigma m}(\varrho)g_{\sigma'' m''}(\varrho)g_{\sigma' m'}(\varrho')g_{\sigma''' m'''}(\varrho')\cdot$$

$$\cdot\,\delta_{L,K-K''}\delta_{L,K'''-K'}\delta_{n,m-m''}\delta_{n,m'''-m'}\,,$$

G contains factors $\delta_{m+m',m''+m'''}\delta_{K+K',K''+K'''}$.

We now introduce the unitary transformation

$$W_1 = \exp\left[c_{00\mu}a^+_{00\mu} - c^*_{00\mu}a_{00\mu}\right] \tag{4.4}$$

expressing the privileged role of the single particle state. Then

$$W_1 a_{00\mu}W_1^{-1} = c_{00\mu} + a_{00\mu}\,. \tag{4.5}$$

For simplicity we write $c_{00\mu} = c$ and $a_{00\mu} = a_\sigma$.

The transformed Hamiltonian is expressed in the form

$$W_1\mathscr{H}W_1^{-1} = \sum_{i=0}^{4}\mathscr{H}_i\,, \tag{4.6}$$

with

$$\mathscr{H}_0 = \left((00\mu\,|\,T\,|\,00\mu) - E - \hbar\omega\mu\right)|c|^2 + (00\mu;\,00\mu\,|\,G\,|\,00\mu;\,00\mu)\,|c|^2|c|^2\,,$$

$$\mathscr{H}_1 = \sum{}'\{(0\sigma\mu\,|\,T\,|\,00\mu) - (E+\hbar\omega\mu)\}a^+_\sigma + \text{h. c.} +$$

$$+ \sum\{(0\sigma\mu;\,00\mu\,|\,G\,|\,00\mu;\,00\mu) + (00\mu;\,0\sigma\mu\,|\,G\,|\,00\mu;\,00\mu)\}a^+_\sigma c^* c^2 + \text{h. c.}\,,$$

$$\mathscr{H}_2 = \sum\{(K\sigma m\,|\,T\,|\,K'\sigma' m') - (E+\hbar\omega m)\delta_{\sigma\sigma'}\}a^+_{K\sigma m}a_{K\sigma' m} +$$

$$+ 2|c|^2\sum\{(K\sigma m;\,00\mu\,|\,G\,|\,K\sigma' m;\,00\mu) + (K\sigma m;\,00\mu\,|\,G\,|\,00\mu;\,K'\sigma' m')\}a^+_{K\sigma m}a_{K\sigma' m} +$$

$$+ c^2\sum(K\sigma m;\,-K\sigma',\,2\mu-m\,|\,G\,|\,00\mu;\,00\mu)a^+_{K\sigma m}a_{-K,\sigma',\,2\mu-m} + \text{h. c.}\,,$$

$$\mathscr{H}_3 = 2c\sum(K\sigma m;\,K'\sigma' m'\,|\,G\,|\,K+K',\,\sigma'',\,m+m+m'-\mu;\,00\mu)a^+_{K\sigma m}a_{K'\sigma' m'}\cdot$$

$$\cdot\,a_{K+K',\sigma'',m+m'-\mu} + \text{h. c.}\,,$$

$$\mathscr{H}_4 = H_4\,.$$

In the limit of weakly interacting bosons $|c_{00\mu}|^2$ is N, but for finite interactions there is a depletion effect which reduces the value to a finite fraction of N. In the limit of weak interactions the function is determined by the

condition that $\mathscr{H}_1$ vanishes. There will then be no constant term in the equation of motion of the $a_{K\sigma m}$. This leads to the requirement

$$(4.7)\quad -\frac{\hbar^2}{2M}\nabla^2\varphi_{00\mu}+|c|^2\varphi_{00\mu}(\boldsymbol{x})\int V(\boldsymbol{x}-\boldsymbol{y})\,|\varphi_{00\mu}(\boldsymbol{y})|^2\,\mathrm{d}^3y=(E+\hbar\omega\mu)\varphi_{00\mu}(\boldsymbol{x})\,.$$

This is substantially the same as eq. (2.4) of the semiclassical theory. It is the same condition as $\delta\mathscr{H}_0/\delta\varphi^*_{00\mu}=0$ for fixed $|c|^2$, subject to the multiplier conditions

$$\int|\varphi_{00\mu}|^2\,\mathrm{d}^3x=1\,,\qquad i\hbar\int\varphi^*_{00\mu}\frac{\partial\varphi_{00\mu}}{\partial\vartheta}\,\mathrm{d}^3x=\hbar\mu\,.$$

The operator which takes the place of the L^μ of eq. (3.21) is

$$(4.8)\qquad L^\mu=-\frac{\hbar^2}{2M}\nabla^2+|c|^2\int V(|\boldsymbol{x}-\boldsymbol{y}|)\,|\varphi_{00\mu}(y)|^2\,\mathrm{d}^3y-(E+\hbar\omega\mu)\,.$$

If we use the eigenfunctions and eigenvalues $\varphi_{K\sigma m}$, $E_{K,\sigma}$ of L^μ, and multiply

$$(4.9)\qquad L^\mu\varphi_{K\sigma m}=E_{K\sigma}\varphi_{K\sigma m}\equiv\left(E^\mu_\sigma-E+\hbar\omega\mu+\frac{\hbar^2K^2}{2M}\right)\varphi_{K\sigma m}\,,$$

by $\varphi^*_{K'\sigma'm'}$ and integrate over space, we find

$$(4.10)\quad \langle K'\sigma'm'\,|\,T\,|\,K\sigma m\rangle-(E+\hbar\omega m)\,\delta_{K,K'}\delta_{\sigma,\sigma'}\delta_{m,m'}+$$
$$+\,2\,|c|^2(K'\sigma'm';\,00\mu\,|\,G\,|\,K\sigma m;\,00\mu)=E_{K\sigma}\delta_{K,K'}\delta_{\sigma,\sigma'}\delta_{m,m'}\,.$$

$\mathscr{H}_2$ then takes the simple form

$$(4.11)\quad \mathscr{H}_2=\sum E_{K\sigma}a^+_{K\sigma m}a_{K\sigma m}+2\sum|c|^2(K\sigma m;\,00\mu\,|\,G\,|\,00\mu;\,K\sigma' m)\,a^+_{K\sigma m}a_{K\sigma' m}+$$
$$+\sum(K\sigma m;\,-K,\,\sigma',\,2\mu-m\,|\,G\,|\,00\mu;\,00\mu)c^2a^+_{K\sigma m}a_{-K,\sigma',2\mu-m}+\text{h. c.}$$

The main point is the strong coupling of $a_{K,\sigma m}$ to $a_{-K,\sigma,a\mu-m}$. We may again partially diagonalize $\mathscr{H}_2$ (neglecting the coupling of the radial modes) by introducing the normal mode transformation W_2.

$$(4.12)\quad\begin{cases}W_2a_{K,\sigma,m}W_2^{-1}\equiv a'_{K,\sigma,m}=\cosh\gamma\,a_{K\varrho m}+\sinh\gamma\,a^+_{-K,\sigma,2\mu-m}\,,\\ W_2a^+_{-K,\sigma,2\mu-m}W_2^{-1}\equiv a'^+_{-K,\sigma,2\mu-m}=\sinh\gamma\,a_{K,\sigma,m}+\cosh\gamma\,a^+_{-K,\sigma,2\mu-m}\,,\\ \gamma=\gamma_{K,\sigma,m}=\gamma_{-K,\varrho,2\mu-m}\,.\end{cases}$$

With

$$\text{(4.13)}\qquad \begin{cases} \mathscr{E}_{K\sigma m} = E_{K\sigma m} + 2\,|c|^2(K\sigma m;\ 00\mu\,|\,G\,|\,00\mu;\ K\sigma m)\,, \\ \lambda_{K\sigma m} = \varepsilon_{K\sigma m}/c^2(K\sigma m;\ -K\sigma,\ 2\mu - m\,|\,G\,|\,00\mu;\ 00\mu)\,, \end{cases}$$

we find for $\gamma_{K\sigma m}$

$$\text{(4.14)}\qquad (1 + \operatorname{tgh}^2\gamma_{K\sigma m}) + \lambda_{K\sigma m}\operatorname{tgh}\gamma_{K\sigma m} = 0\,.$$

The zero point energy is

$$\text{(4.15)}\qquad \varepsilon_0 = \sum_{K\sigma m} \mathscr{E}_{K\sigma m}\left\{\sinh^2\gamma_{K\sigma m} + \frac{1}{\lambda_{K\sigma m}}\sinh 2\gamma_{K\sigma m}\right\},$$

while the spectrum of excitations is

$$\text{(4.16)}\qquad \sum_{K\sigma m} a^+_{K\sigma m} a_{K\sigma m}\varepsilon_{K\sigma m}\,,$$

where

$$\varepsilon_{K\sigma m} = 2\,\mathscr{E}_{K\sigma m}\left(\cosh^2\gamma_{K\sigma m} + \frac{\sinh^2 2\gamma_{K\sigma m}}{\lambda_{K\sigma m}}\right).$$

Thus, we have

$$\text{(4.17)}\qquad \begin{cases} W_2\mathscr{H}_2 W_2^{-1} = \varepsilon_0 + \sum a^+_{K\sigma m} a_{K\sigma m}\varepsilon_{K\sigma m} + \mathscr{H}_r\,, \\ \mathscr{H}_r = 2\sum\limits_{\sigma\neq\sigma'} |c|^2(K\sigma m;\ 00\mu\,|\,G\,|\,00\mu;\ K\sigma' m)(a^+_{K\sigma m}\, a_{K\sigma' m})' + \\ \qquad + \sum\limits_{\sigma\neq\sigma'} c^2(K\sigma m;\ K\sigma',\ 2\mu - m\,|\,G\,|\,00\mu;\ 00\mu)(a^+_{K\sigma m} a^+_{K\sigma',2\mu-m})' + \text{h. c.} \end{cases}$$

In the lowest approximation, neglecting depletion effects and the modifications of the self consistent potential arising from close collisions, $|c_{00\mu}|^2 = N$ and the Lagrange multiplier ω is zero. The results are then exactly the same as for the semi-classical theory, and mainly provide a formal justification for adding the shift zero-point energy of the oscillations to the self-consistent field energy of the vortex state. The state vectors in this approximation are $\Psi_{\text{new}} = W_1 W_2 \Phi_{\text{old}}$, where Φ_{old} runs through a set of states consisting of a vacuum state Φ_0 such that $a_{K\sigma m}\Phi_0 = 0$, and a set of states obtained by operating separately on Φ_0 with the creation operators

$$a^+_{K\sigma m} = \int \psi^+(x)\varphi_{K\sigma m}(x)\,\mathrm{d}^3x\,,$$

$$\Psi_{\text{new}} = \exp\left[\sum_\sigma (c a^+_\sigma - c^* a_\sigma)\right]\cdot\exp\left[\sum \gamma_{K\sigma m}(a^+_{K\sigma m} a^+_{K,\sigma,2\mu-m} - \text{h. c.})\right]\cdot\Phi_{\text{old}}\,.$$

It is of some interest to examine the expectation values of physical quantities to see the modifications of the self consistent field approximation induced by the zero-point motions of the field. The expectation value of the density is

$$\begin{cases} n(\boldsymbol{x}) = \langle \Psi_0,\ \psi^+(x)\psi(\boldsymbol{x})\Psi_0\rangle = \langle \Phi_0,\ (W_1W_2)^{-1}\psi^+\psi(W_1W_2)\Phi_0\rangle \\ \qquad = |\varphi_{00\mu}(\boldsymbol{x})|^2|c|^2 + \sum \sinh^2\gamma_{K\sigma m}|g^\mu_{\sigma m}(\varrho)|^2 . \end{cases} \tag{4.18}$$

The main interest is when $\varrho \to 0$. For $m \neq 0$ all terms tend to zero. The only contribution is from $m = 0$ and is $\sum_{K\sigma} \sinh^2\gamma_{K\sigma 0}|g^\mu_{\sigma 0}(\varrho)|^2 \neq 0$. The zero-point fluctuations of the s-wave phonons give a non-zero density at the vortex line. We may define the expectation value of the velocity as

$$\overline{v_\vartheta} = \frac{1}{n(x)}\frac{i\hbar}{2}\left\langle \Psi_0, \left(\psi^+\frac{1}{\varrho}\frac{\partial\psi}{\partial\vartheta} - \frac{1}{\varrho}\frac{\partial\psi^+}{\partial\vartheta}\psi\right)\Psi_0\right\rangle . \tag{4.19}$$

The density is now non-zero at $\varrho = 0$. But the numerator is

$$|c_{00\mu}|^2\frac{\hbar\mu}{\varrho}|g^\mu_{0\mu}(\varrho)|^2 + \sum \sinh^2\gamma_{K\sigma m}\frac{\hbar m}{\varrho}|g^\mu_{\sigma m}(\varrho)|^2 . \tag{4.20}$$

It tends to zero as $\varrho \to 0$, since the $m = 0$ term contains a factor $\hbar m$, and is thus absent from the sum. Hence, the expectation value of the velocity is finite, and is in fact zero at $\varrho = 0$. For $\mu = 1$

$$\overline{v_\vartheta} \xrightarrow[\varrho\to 0]{} \frac{|c_{001}|^2}{n(\varrho = 0)}\frac{|g'_{01}(\varrho)|^2}{\varrho} , \tag{4.21}$$

since $g'_{01}(\varrho)$ goes as $\alpha j_1(\sqrt{(2ME/\hbar^2)}\varrho)$, where α is a normalization factor, we have

$$\overline{v_\vartheta} \xrightarrow[\varrho\to 0]{} \frac{|c_{001}|^2}{n(\varrho = 0)}\hbar\varrho\left(\frac{2ME}{\hbar^2}\right)\alpha^2 . \tag{4.22}$$

Thus the vorticity $\overline{w} = \frac{1}{2}(1/\varrho)(\partial/\partial\varrho)(\varrho\overline{v_\vartheta})$ has the limiting value

$$\overline{w} \xrightarrow[\varrho\to 0]{} 2\frac{\overline{v_\vartheta}}{\varrho} \to \frac{2|c_{001}|^2}{n(\varrho = 0)}\hbar\alpha^2\left(\frac{2ME}{\hbar^2}\right) = \text{constant} \neq 0 . \tag{4.23}$$

The vorticity has a finite value in the core.

It is clear that these results depend on the general form of the vortex wave-function $W_1W_2\Phi_0$, and are independent of the precise values of $|c_{00\mu}|^2$ and $\gamma_{K\sigma m}$. As described in ref. ([4b]), in a higher approximation these parameters are shifted from the values given by the theory that considers only $\mathscr{H}_0+\mathscr{H}_1+\mathscr{H}_2$. One must consider the modifications induced when the creation and annihilation operators are reordered after the normal mode transformation. This

includes a diagonal contribution to the ground state energy from $W_2\mathscr{H}_4W_2^{-1}$. The best ground state for the chosen type of approximate state vector is obtained variationally. The parameters $c_{00\mu}$, ω, and E, are determined by minimizing the ground state energy expectation value with respect to $c^*_{00\mu}$, and imposing the two Lagrange multiplier conditions. Since we are not working with a plane wave basis, the reordering of $W_2\mathscr{H}_3W_2^{-1}$ leads to a modified self-consistent potential because terms linear in the creation and annihilation operators appear. $W_2\mathscr{H}_4W_2^{-1}$ also contains terms that lead to a modified normal mode problem, *i.e.* a redetermination of the $\gamma_{K\sigma m}$. Following these lines we may obtain an improved value for the energy of a vortex state without phonons. One main effect is the replacement of the total density N/Ω by the superfluid density $|c_{00\mu}|^2/\Omega$ in the estimates. Further improvements must consider a more general normal mode transformation to take into account the radial mode coupling, and also off diagonal contributions of $W_2(\mathscr{H}_3+\mathscr{H}_4)W_2^{-1}$. It would be worthwhile to undertake a systematic calculation to give a clear development of all quantities in the strength of the interparticle potential. A trustworthy calculation for the actual case of liquid helium would be difficult. It is likely that the zero-point oscillation-spread of the core density is comparable to the self-consistent field contribution. It is well known that the Bogoljubov normal mode transformation is too restricted for studying the phonon spectrum in higher approximations. The off diagonal elements of $W_2(\mathscr{H}_3-\mathscr{H}_4)W_2^{-1}$ must be considered to avoid spurious energy gaps.

It is, however, not our concern here to study these refinements. Further development might be preferably undertaken by other methods [7-10]. The main point, however, already appears in the simpler treatment presented here. It is the specification of a special, highly occupied, « vortex » single-particle state $\varphi_{00\mu}(\boldsymbol{x})$. For weakly repelling bosons, where the self-consistent field contribution to the structure of the vortex dominates quantum fluctuation effects, the picture is particularly clear.

The wave function representing an elementary vortex state has an expectation value $N\hbar\mu$ for the angular momentum. The projected state with the exact $N\hbar\mu$ is of course orthogonal to the ground state without a vortex. In our approximation the orthogonality occurs as $N\to\infty$, essentially because the inner product

$$\left\langle \exp\left[f_0\int\psi^+(x)\,\mathrm{d}^4x-\text{c.c.}\right]\Phi_0\middle/\exp\left[\int f''(\boldsymbol{x})\psi^+(\boldsymbol{x})\,\mathrm{d}^3x-\text{c.c.}\right]\Phi_0\right\rangle\to\exp\left[-N\beta\right].$$

(7) K. A. Brueckner and K. Sawada: *Phys. Rev.*, **106**, 1117 (1957).
(8) S. T. Belyaev: *Soviet Physics JETP*, **7**, 289 (1958).
(9) R. Abe: *Prog. Theor. Phys.*, **19**, 1, 57, 407, 699, 713 (1958).
(10) N. M. Hugenholtz and D. Pines: *Phys. Rev.*, **116**, 489 (1959).

Cf. the similar discussion of the periodic ground states in ref. ([4b]). This argument holds when we consider corresponding states with a small number of excitations. But it breaks down when there are of the order of N excitations, for example at a finite temperature near the λ-point. Furthermore, the terms $W_2\mathscr{H}_3W_2^{-1}$ and $W_2\mathscr{H}_4W_2^{-1}$ create and annihilate only a small number of particles in a single act. Thus an elementary vortex can only decay by a very high order process in perturbation theory, *i.e.* it has an essentially macroscopic lifetime. We do not try to estimate this lifetime here. With a large number of excitations the vortex state becomes seriously depleted and the core grows in size and fluctuates. There are then rapid transitions to a circulation-free state with production of phonons. The problem appears difficult to discuss precisely from the present point of view. We have an overcompleteness of states characteristic of self-consistent field theories. Each $f^\mu(\boldsymbol{x})$ and the associated excitation spectrum presumably spans the same space of state vectors. The approximate wave functions can then be accurate only when are there is a small number of excitations.

5. – Further discussion of the semiclassical theory.

The equation of motion for the classical field $\psi(\boldsymbol{x}, t)$ may be rewritten in hydrodynamic form. Putting $\psi = E\exp[iS/\hbar]$, where R and S are real, one finds

$$\frac{\partial}{\partial t}(R^2) = -\operatorname{div}\left(R^2\frac{\nabla S}{M}\right), \tag{5.1}$$

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2M} - \frac{\hbar^2}{2M}\frac{\nabla^2 R}{R} + \int V(\boldsymbol{x}-\boldsymbol{y})R^2(\boldsymbol{y})\,\mathrm{d}^3y\,. \tag{5.2}$$

We introduce the velocity field $\boldsymbol{v} = \nabla S/M$ and take the gradient of the second equation. Then

$$M\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) + \nabla\Pi = 0\,, \tag{5.3}$$

Π is given by the functional

$$\Pi = -\frac{\hbar^2}{2M}\frac{\nabla^2 R}{R} + \int V(\boldsymbol{x}-\boldsymbol{y})R^2(\boldsymbol{y})\,\mathrm{d}^3y\,. \tag{5.4}$$

consisting of a « quantum » contribution and a contribution arising from particle interactions. It has the dimensions of energy. In the usual hydrodynamics one assumes the pressure p is a function of the density and $\Pi = \int \mathrm{d}p/n$. But the present hydrodynamics is quite different. It is useful to introduce

the density $n = R^2$ and to write (summation convention)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_k}(n v_k) = 0 , \tag{5.6}$$

$$M \frac{\partial}{\partial t}(n v_i) + M \frac{\partial}{\partial x_k}(n v_i v_k) = n \frac{\partial \Pi}{\partial x_i} , \tag{5.7}$$

$n(\partial \Pi / \partial x_k)$ can be expressed as [11]

$$n \frac{\partial \Pi}{\partial x_k} = - n \frac{\partial}{\partial x_k} \int V(x - y) n(y) \, \mathrm{d}^3 y + \frac{\partial \sigma_{ik}}{\partial x_k} , \tag{5.8}$$

with a stress tensor

$$\sigma_{ik} = \frac{\hbar^2}{2M} n \frac{\partial^2 (\log n)}{\partial x_i \partial x_k} ; \qquad \sum \frac{\partial \sigma_{ik}}{\partial x_k} = n \frac{\partial}{\partial x_i} \left(\frac{\hbar^2}{2M} \frac{\nabla^2 R}{R} \right) . \tag{5.9}$$

This differs from ordinary hydrodynamics in that the stress depends on derivatives of the density, rather than velocity.

The elementary line vortex has the property that $\nabla^2 S = 0$ and $\nabla S \cdot \nabla R = 0$, and that $R = 0$ on the vortex line. The quantum pressure $-(\hbar^2/2M)(\nabla^2 R/R) \to$ $\to - \infty$ as $\varrho \to 0$ and cancels $(\nabla S)^2/2M$ which $\to + \infty$, leaving a finite quantity. It is clear since the vortex core is small in extent, that we can generalize the argument, and find approximate solutions representing steady patterns of vortices separated by distances greater than the core diameter. One way is to look first for solutions of $\nabla^2 S = 0$, *e.g.* ring vortices, sets of line vortices, etc. Then one finds the appropriate behaviour of R near the lines of singularity. This brings into play the type of classical hydrodynamic argument already used in interpreting experimental properties of superfluid helium. The main role of the quantum mechanics, at least for weakly interacting bosons, is to provide a foundation for this procedure, and to ensure that there is a definite theory of the structure of the vortex core. This type of consideration is foreign to classical (even compressible) hydrodynamics. The structure depends on the quantum pressure term, as is evident from the fact that the characteristic size depends on a De Broglie wavelength. In addition, the study of the quantum state vectors associated with a flow pattern should show that decay is possible only in a macroscopic time.

Formally, each solution of the hydrodynamic equations defines a basis function which is occupied by a finite fraction of the particles. Such a state is not as symmetrical as the elementary cylindrical vortex and would probably not be considered from the point of view of ordinary quantum mechanics. The complete set of functions orthogonal to the basis function permits an

[11] T. Takabayasi: *Prog. Theor. Phys.*, **8**, 143 (1952); **9**, 187 (1953).

analysis of the small oscillation spectrum comparable to that of Sections 3 and 4.

Phenomenological single fluid hydrodynamic equations have been used to study processes such as scattering of phonons and protons from vortices ([12]). These theories assume a pressure density relation such that $(\mathrm{d}p/\mathrm{d}\varrho)^{\frac{1}{2}}$ agrees with the observed first sound velocity. In contrast the present theory contains the quantum pressure term and is directly related to basic quantum mechanics. However, the defect of our procedure is that only weakly interacting particles can be treated in a clear manner. It should be possible, in the spirit of a pseudopotential method, of Brueckner's ideas, or of Landau's theory of a Fermi liquid to justify replacing the interaction potential *ab initio* to treat strong interactions. Then the semiclassical theory itself would become useful for quantitative applications to superfluid helium.

* * *

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([12]) L. P. Pitaevski: *Soviet Physics JETP*, **8**, 888 (1958).

RIASSUNTO (*)

Sviluppiamo una teoria delle eccitazioni dei vortici lineari elementari, per un sistema di bosoni debolmente repulsivi. Lo stato di vortice è caratterizzato dalla presenza di una frazione finita di particelle nello stato di particella singola con momento angolare intero. La dipendenza radiale dello stato densamente occupato segue da una equazione di campo autocongruente. La funzione radiale e la densità di particelle associata sono essenzialmente costanti dovunque tranne che nell'interno di un nocciolo, dove si annullano. La dimensione del nocciolo è uguale alla lunghezza d'onda di de Broglie associata con l'energia media di interazione per particella. Il valore previsto per la velocità ha la dipendenza radiale del vortice classico. In questa approssimazione di Hartree la vorticità è nulla dovunque tranne che sulla linea di vortice. Quando si raffina la descrizione dello stato sino ad includere le oscillazioni nel punto zero del campo fononico, la vorticità si estende a tutto nocciolo. Questi risultati confermano nei punti essenziali gli argomenti intuitivi di Onsager e Feynman. I fononi che si muovono normalmente alla linea di vortice hanno eccitazioni congruenti con quantità di moto uguale e contraria rispetto al substrato di particelle in movimento che costituisce il vortice. Il movimento del vortice risolve la degenerazione dei fononi di Bogoljubov rispetto al numero quantico azimutale.

(*) *Traduzione a cura della Redazione.*

Conditions for Bound States with a Given Set of Interaction Poles in Chew-Mandelstam Theory.

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(ricevuto il 17 Gennaio 1961)

Summary. — Using the potential model with the left-hand cut approximated by n poles, conditions are derived under which no extra « bound state » poles need be introduced. The procedure adopted is to find the domain in n-space convered by the sets of values of residues in which no extra poles need be introduced. It is shown that any point in this domain can be reached by adjusting only n poles on the unphysical sheet.

1. – Introduction.

In the Chew-Mandelstam approach (1), a partial wave amplitude A for a scattering process is expressed as a function of ν, the square of the barycentric three-momentum k, and has two cuts in the ν plane (both on the real axis) from $-\infty$ to ν_L ($\nu_L < 0$) and from 0 to ∞, respectively, plus possible additional poles on the real axis. There may also be some circular cuts, but these disappear if sufficient masses are taken to be equal, and we shall assume this condition satisfied.

A is then expressed as $N(\nu)/D(\nu)$, $N(\nu)$ having only the left-hand cut and $D(\nu)$ the right-hand cut. The imaginary part of the reciprocal amplitude is known along the right-hand cut, for example if all particles have unit mass,

$$\mathrm{Im}\,(A)^{-1} = -\frac{\nu}{\nu+1}R\,,$$

(1) A summary of the dynamics is given by G. F. Chew: *Double Dispersion Relations* U.C.R.L. 9289 (June 1960), along with further references.

where R is the ratio of total to elastic partial wave cross-section. The unknown in the problem is the discontinuity $f(\nu)$ across the left-hand cut. $f(\nu)$ can be regarded as a dynamical description of the problem, and the method of solution is to guess $f(\nu)$ and then solve the coupled integral equation in N, D arising from Cauchy's formula by an iterative procedure.

Recently several authors have tackled the problem by replacing the left-hand cut with a series of poles ([2]), and it is sometimes found that further poles, usually interpreted as representing bound states, must be introduced on the negative axis if the equations are to be consistent. These extra poles could have been fed in initially, in which case no further poles would have arisen, and in this case the left-hand axis singularities can be regarded as furnishing a complete dynamical description of the scattering system. We shall call the initial information consistent if not further left-hand axis singularities need be introduced to make the equations consistent, that is to complete the dynamical description. It must be emphasized that we are not for the moment concerned with the Castillejo-Dalitz-Dyson ([3]) procedure, which amounts to adding extra poles to an already consistent set.

A simple example of a consistency condition is given by replacing the left-hand cut with a single « interaction » pole of small residue. On increasing the residue, which corresponds to strengthening the attractive potential well represented by the pole, a critical value is reached beyond which a further pole must appear on the physical sheet, that is, if the potential is sufficiently strong, a bound state will appear.

In order to discover the kind of conditions for consistency which can arise, the model of potential theory is taken. The reason for doing this is that if the left-hand cut is a series of poles, then

$$A = \frac{1}{(P(\nu)/Q(\nu)) - i\sqrt{\nu}},$$

in the low energy approximation, where P, Q are polynomials. The scattering matrix S is given by

$$S = 1 + 2ikf$$

and $f = A$ in the low energy limit, so

(1.1) (i)
$$S = \frac{P + ikQ}{P - ikQ} = \frac{h(k)}{h(-k)},$$

([2]) See *e.g.* J. W. Moffatt: *R.I.A.S.*, preprints (1960).

([3]) L. Castillejo, R. H. Dalitz and F. J. Dyson: *Phys. Rev.*, **101**, 453 (1956).

which is exactly analogous to the Jost form for potential scattering [1]. The other conditions on $S(k)$ we use are:

ii) $$S(-k^*) = S(k)^* .$$

iii) Causality: all singularities of $S(k)$ are on the imaginary axis or at complex conjugate points in the lower half plane.

Poles of A correspond to poles of S, and it is more convenient to work with S. k is used rather than k^2 as variable, since in potential theory the right-hand cut is simply a square root cut. This is only true in field theory up to the first production threshold.

In this paper explicit conditions for consistency are derived in the two three-pole cases and it is indicated how the methods employed can be extended without difficulty of principle to any higher number of poles. The basis of the method is to fix the positions of the poles and find the domain in n-dimensional space covered by the sets of values of their residues as a number of poles are adjusted on the unphysical sheet. A further result, which follows from the mathematical structure of the problem, is that any point in the domain can be reached by adjusting n poles on the unphysical sheet, and addition of any further enumerable number of poles to the unphysical sheet does not extend the domain.

2. – The two-pole case.

2·1. *Boundaries of the domain.* – We work with $\varkappa = ik$, since all singularities in the k-plane are real or conjugate complex in pairs. If the left-hand cut is replaced by two poles at $k = -l$, $k = -m$ and there are two poles in the right-hand (unphysical) half plane at (x, y) with x, y real or conjugate complex, then from (1.1)

$$S(\varkappa) = \frac{\varkappa - l}{\varkappa + l} \cdot \frac{\varkappa - m}{\varkappa + m} \cdot \frac{\varkappa + x}{\varkappa - x} \cdot \frac{\varkappa + y}{\varkappa - y} . \tag{2.1}$$

The residue at $\varkappa = -l$ is

$$\begin{cases} R_l = 4\pi i l \left(\dfrac{l+m}{l-m}\right) \lambda , & \lambda = \left(\dfrac{l-x}{l+x}\right)\left(\dfrac{l-y}{l+y}\right), \\ R_m = 4\pi i m \left(\dfrac{m+l}{m-l}\right) \mu , & \mu = \left(\dfrac{m-x}{m+x}\right)\left(\dfrac{m-y}{m+y}\right). \end{cases} \tag{2.2}$$

[1] R. JOST: *Helv. Phys. Acta*, **20**, 256 (1947).

We define $D(l, m)$ as the domain in the (λ, μ) plane given by $x+y$, $xy > 0$. $D(l, m)$ lies entirely inside the square $|\lambda| \leqslant 1$, $|\mu| \leqslant 1$ although it includes (1, 1) and (−1, −1) given by $x = y = 0$, $x = 0$, $y = \infty$. Its boundaries are given by two possibilities:

i) $x = 0$, *i.e.* S has a pole at the origin, and it is possible to make an arbitrarily small change in (λ, μ) so that the pole appears on the physical sheet (this is because the Jacobian $\partial(\lambda, \mu)/\partial(x, y)$ is non-zero, except at the point (1,1)). $x = 0$ gives

$$\lambda = \frac{l - y}{l + y}, \qquad \mu = \frac{m - y}{m + y}, \qquad 0 \leqslant y \leqslant \infty,$$

and eliminating y,

$$C_{0y} \equiv (\lambda\mu - 1)(l - m) + (l + m)(\lambda - \mu) = 0 . \tag{2.3}$$

This is a rectangular hyperbola with horizontal and vertical asymptotes through (1, 1) and (−1, −1) and the segment of the curve between these two point forms part of the boundary of $D(l, m)$. It cuts $\mu = 0$ at $\lambda = (l - m)/(l + m)$

ii) $x = \infty$. In this case a small change can be made in (λ, μ) so that either:

a) x goes through ∞ and appears at a large distance from the origin on the left-hand axis, representing a short range interaction potential, or

b) if we are willing to introduce further poles, x can remain at ∞ and an extra pole can be moved into the physical region from the origin, leaving a further pole at the origin.

The boundary curve in this case is

$$C_{\infty y} \equiv (\lambda\mu - 1)(l - m) + (l + m)(\mu - \lambda) = 0 \tag{2.4}$$

just the reflection in the origin of the curve for $x = 0$. These two curves specify completely the boundaries of $D(l, m)$ but it is of some interest to find the areas of the domain covered by the two possible cases of x, y real and x, y conjugate complex. The boundary line is given by $x = y$, or

$$\lambda = \left(\frac{l - x}{l + x}\right)^2, \qquad \mu = \left(\frac{m - x}{m + x}\right)^2, \qquad 0 \leqslant x \leqslant \infty.$$

By analogy with C_{0y}, this can be written

$$C_{xx} \equiv (\sqrt{\lambda\mu} - 1)(l - m) + (1 + m)(\sqrt{\lambda} - \sqrt{\mu}) = 0 .$$

The curves have the form shown in Fig. 1.

It is clear that the values of λ, μ given by $x = \overline{y}$ must lie inside C_{xx}, since the origin is given by $(x, y) = (l, m)$.

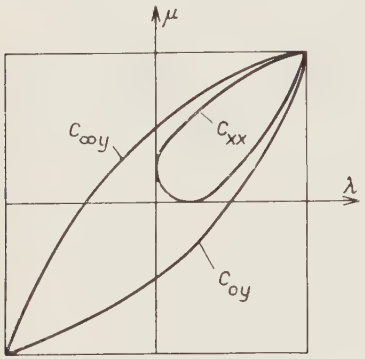

Fig. 1.

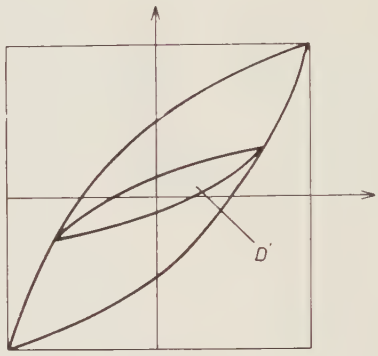

Fig. 2.

2·2 *Extra unphysical poles.* – The addition ot two extra poles at z, t on the unphysical sheet gives

$$\lambda = \left(\frac{l-x}{l+x}\right)\left(\frac{l-y}{l+y}\right)\left(\frac{l-z}{l+z}\right)\left(\frac{l-t}{l+t}\right) = \lambda'(x, y)\,\lambda''(z, t)\,,$$

so $\lambda = \lambda'\lambda''$, $\mu = \mu'\mu''$ where $(\lambda', \mu') \in D(l, m)$ and $(\lambda'', \mu'') \in D(l, m)$. We now prove that this implies $(\lambda, \mu) \in D(l, m)$.

The worst possible case for a given λ'/μ' is the point on the boundary in this direction. Varying (λ'', μ'') over $D(l, m)$ $(\lambda'\lambda'', \mu'\mu'')$ covers a domain D' given by scaling down $D(l, m)$ in the ratios λ', μ' in the two directions (see Fig. 2).

As the equations of the boundary curves are known, it is easy to verify that $D' \in D(l, m)$ and it follows by induction that any number of extra poles will not extend the domain covered by (λ, μ) beyond $D(l, m)$.

3. – The three-pole case.

3·1. *The domain* $D(l, m, n)$. – Taking the physical poles at $-l$, $-m$, $-n$ we have to find $D(l, m, n)$ defined by

$$\lambda = \frac{l-x}{l+x}\cdot\frac{l-y}{l+y}\cdot\frac{l-z}{l+z}\,,$$

$$\mu = \frac{m-x}{m+x}\cdot\frac{m-y}{m+y}\cdot\frac{m-z}{m+z}\,,$$

$$\nu = \frac{n-x}{n+x}\cdot\frac{n-y}{n+y}\cdot\frac{n-z}{n+z}\,.$$

$x, y, z > 0$ all real or one real, two complex conjugates.

The domain lies inside the cube $(\pm 1, \pm 1, \pm 1)$ and includes the points $(1, 1, 1)$ and $(-1, -1, -1)$. Its intersection with $\lambda = 0$ is given by $x = l$ so (μ, ν) lies in a domain identical with $D(l, m)$ scaled down by different factors in the x- and y-directions. Hence $D(l, m, n)$ is rather like a banana, with two « edges », stretching from $(1, 1, 1)$ to $(-1, -1, -1)$.

3·2. *Formal derivation of* $D(l, m, n)$. – The curve C_x is defined by the parametric equations

$$\lambda = \frac{l - x}{l + x}, \qquad \mu = \frac{m - x}{m + x}, \qquad \nu = \frac{n - x}{n + x}.$$

As x varies from 0 to ∞, (λ, μ, ν) moves along C_x from $(1, 1, 1)$ to $(-1, -1, -1)$, and during this variation (λ, μ) traces out C_{x0} (see eq. (2.3)), hence C_x lies on a rectangular hyperboloid cylinder parallel to the ν axis, and similarly lies on such cylinders parallel to the other axes (see Fig. 3).

If both x and y are varied, keeping $z = 0$, a surface is generated, which we denote by C_{xy}. Some idea of its shape is given by taking a point x_0 or C_x and then varying y from 0 to ∞. (λ, μ, ν) then describes a curve which is just C_x scaled down proportionally to the co-ordinates of the point x_0, and going from this point to its reflection in the origin. Varying x_0 maps out the whole surface (except for that part given by $y = \bar{x}$, but this adds no complications).

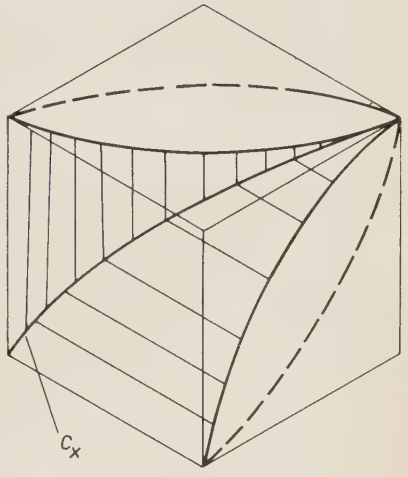

Fig. 3.

Another method of deriving C_{xy}, giving a clearer idea of its form is explained in Appendix A.

Putting $z = \infty$ gives the surface $-C_{xy}$, the reflection of C_{xy} in the origin, and C_{xy}, $-C_{xy}$ meet along the two curves C_x, $-C_x$ thus forming a closed surface, enclosing a domain Δ. Denoting by $C_{x_0 y_0 z}$ the curve by fixing (x_0, y_0) on C_{xy} then varying z from 0 to ∞, it is shown in Appendix B that $C_{x_0 y_0 z}$ lies entirely inside Δ for all $x_0, y_0 \geqslant 0$. From this it follows that Δ is identical with $D(l, m, n)$, because the only possible boundaries of $D(l, m, n)$ are given by one pole at least being at 0 or ∞, *i.e.* C_{xy} and $-C_{xy}$, or by the Jacobian being zero *i.e.* along the surface C_{xxy} of curves $C_{x_0 x_0 y}$ which *a fortiori* all lie inside Δ, and as $C_{xy} \in D(l, m, n)$ and all parts of D' are accessible to points of C_{xy} without crossing C_{xxy}, $\Delta \in D(l, m, n)$, so $\Delta \equiv D(l, m, n)$.

3·3. *Extra poles on the unphysical sheet.* – Since all curves of the form $C_{x_0 y_0 z}$ lie inside $D(l, m, n)$ it follows that the surfaces $C_{x_0 yz}$, $-C_{x_0 yz}$ lie inside $D(l, m, n)$. The domain D' enclosed by these surfaces is analogous to that discussed at the end of Section 2, and corresponds to values of (λ, μ, ν) with four poles on the unphysical sheet, one at x_0. Hence by induction as before we find that $D(l, m, n)$ is not extended by including more poles on the unphysical sheet than there are on the physical sheet.

4. – The n-pole case.

The proof in Appendix B that $C_{x_0 y_0 z}$ lies inside $D(l, m, n)$ depends in an inductive fashion on the fact that in the two-dimensional case $C_{x_0 y}$ lies inside $D(l, m)$. It is easy to see that the mathematical structure of the problem is unchanged if we add an extra dimension, thus $C_{x_0 y_0 z_0 t}$ lies inside $D(l, m, n)$ in the four-dimensional case by an identical argument. Hence we conclude that in the n-dimensional case only n poles are required on the unphysical sheet to give any point in $D(l, m, n, \ldots)$. It is interesting to note that the only algebra necessary to establish this result is that needed to find $D(l, m, n, p)$ in the three-pole case.

* * *

I should like to thank Professor S. B. TREIMAN for suggseting the problem, and for help and encouragement throughout. I should also like to thank Dr. E. J. SQUIRES, Mr. R. W. LARDNER, Dr. R. J. EDEN and Dr. J. C. POLKINGHORNE for interesting conversations, and the D.S.I.R. for a grant for research.

APPENDIX A

Another method of deriving C_{xy}.

Take

$$\lambda \quad \frac{l - x}{l + x} \cdot \frac{l - y}{l - y},$$

in the form

$$xy(\lambda - 1) + l(x + y)(\lambda + 1) + l^2(\lambda - 1) = 0 ,$$

with similar equations for μ, ν and eliminate xy, $x + y$, 1.

This method must be treated with some caution, as we cannot restrict xy, $x+y$ to be real and positive, but it is helpful in that it gives cross sections of the surface in various ways.

$$D = \begin{vmatrix} \lambda - 1 & l(\lambda+1) & l^2(\lambda-1) \\ \mu - 1 & m(\mu+1) & m^2(\mu-1) \\ \nu - 1 & n(\nu+1) & n^2(\nu-1) \end{vmatrix} = 0 .$$

Putting

$$\alpha = \frac{\lambda-1}{2}, \qquad \beta = \frac{\mu-1}{2}, \qquad \gamma = \frac{\nu-1}{2},$$

$$\Delta = 0 \qquad \text{gives} \qquad \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1,$$

where

$$\alpha = \frac{l(m+n)}{(l-m)(n-l)}, \qquad a+b+c=1 .$$

A section

$$\alpha = \alpha_0 \quad \text{is} \quad \frac{b}{\beta} + \frac{c}{\gamma} = \left(1 - \frac{\alpha}{\alpha_0}\right),$$

i.e. any section of C_{xy} parallel to a co-ordinate plane is part of a rectangular hyperbola with horizontal and vertical axes.

A section $\alpha = k\beta$, k constant, also gives a rectangular hyperbola. This corresponds to a section of C_{xy} on a plane through the line ($\lambda = 1$, $\mu = 1$).

For instance, the section through ($\lambda = 1$, $\mu = 1$) and ($\lambda = 0$, $\mu = 0$) is as shown in Fig. 4.

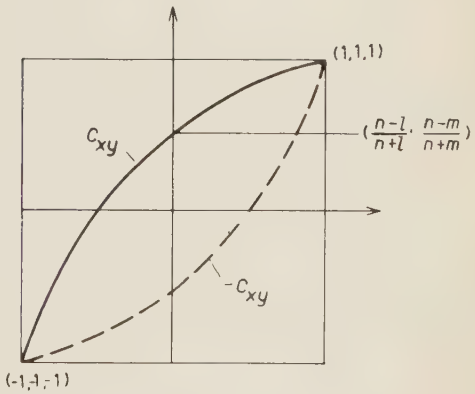

Fig. 4.

Appendix B

Proof that $C_{x_0y_0z}$ lies inside Δ.

It is sufficient to show that $C_{x_0y_0z}$ meets C_{xy} only at the point $(x_0, y_0, 0)$. If (x_0, y_0) gives a point on one of the co-ordinate planes, $C_{x_0y_0z}$ lies in this plane and the problem reduces to the two dimensional case, where $C_{x_0y_0z}$ certainly lies inside C_{xy}, $-C_{xy}$. If there exists some point (x_0, y_0) such that some part of $C_{x_0y_0z}$ lies outside C_{xy}, then if (x_0, y_0) such that some part of $C_{x_0y_0z}$ lies outside C_{xy}, then if (x_0, y_0) is moved continuously along a path to a point on one of the co-ordinate planes, since all curves are continuous and vary

continuously it must reach a point where $C_{x_0y_0z}$ touches the surface at say (x_1, y_1). We shall show that this leads to a contradiction.

The direction ratios for $C_{x_0y_0z}$ are

$$\left(\frac{\partial\lambda}{\partial z}, \quad \frac{\partial\mu}{\partial z}, \quad \frac{\partial\nu}{\partial z}\right),$$

at

$$\lambda = \lambda_1, \qquad \frac{\partial\lambda}{\partial z} = -\frac{2l}{l^2 - z^2}\lambda_1 .$$

If $C_{x_0y_0z}$ touches C_{xy} at $(\lambda_1, \mu_1, \nu_1)$, a point corresponding to parameters (x_1, y_1) then its direction ratios at this point are a linear combination of those for C_{xy_1}, C_{x_1y}, the two curves lying in C_{xy} passing through (x_1, y_1).

The condition for this to be so (assuming none of l, m, n, λ_1, μ_1, $\nu_1 = 0$ since this reduces to the two dimensional case again) is

$$D = \begin{vmatrix} 1 & 1 & 1 \\ l^2 - x_1^2 & m^2 - x_1^2 & n^2 - x_1^2 \\ 1 & 1 & 1 \\ l^2 - y_1^2 & m^2 - y_1^2 & n^2 - y_1^2 \\ 1 & 1 & 1 \\ l^2 - z^2 & m^2 - z^2 & n^2 - z^2 \end{vmatrix} = 0 ,$$

which reduces to

$$(l^2 - m^2)(m^2 - n^2)(n^2 - l^2)(x_1^2 - y_1^2)(y_1^2 - z^2)(z^2 - x_1^2) = 0 ,$$

if all the denominators remain finite.

Taking l, m, n to be distinct, either $x_1 = y_1$ or $z =$ one of x_1, y_1. If $x_1 = y_1$ the curves C_{x_1y}, C_{xy_1} touch at (x_1, y_1) so $D = 0$ does not necessarily mean $C_{x_0y_0z}$ touches C_{xy}. Another set of direction ratios is given by differentiating, which corresponds to subtracting the second row of D from the first, dividing by $x_1 - y_1$ and taking the limit as $x_1 \to y_1$. The new determinant is non-zero unless $z = y_1$.

Hence (x_0, y_0, z), $(x_1, y_1, 0)$ can be written in the form (x_1, x_0, y_0), $(x_1, y_1, 0)$ and taking a two dimensional section this reduces to the case of (x_0, y_0) and $(y_1, 0)$ being identical points, which is only satisfied if $x_0 = y_1$, $y_0 = 0$.

RIASSUNTO (*)

Usando il modello a potenziali con il taglio sinistro approssimato da n poli, si deducono le condizioni per le quali non è necessario introdurre poli extra di « stato legato ». Il procedimento adottato consiste nel trovare il dominio nello n-spazio coperto da gruppi di valori dei residui in cui non è necessario introdurre poli extra. Si mostra che qualunque punto di questo dominio può essere raggiunto sistemando solo n poli sul piano non fisico.

(*) *Traduzione a cura della Redazione.*

Verification at 27 GeV of a Formula for the Determination of the Primary Energy of Jets.

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(ricevuto il 26 Gennaio 1961)

Summary. — Proton interactions in nuclear emulsions at 27 GeV have been analysed in order to verify at this energy the validity of the formula $\lg \gamma_c = \langle \lg | \operatorname{ctg} \theta | \rangle$. Good agreement between expected and experimental values has been found for interactions with $N_h \leqslant 4$ and $n_s \geqslant 4$. The correction to be introduced under the hypothesis of a pion spectrum due to statistical theory is discussed, as well as the dispersion of the value of $\log \gamma_c$ obtained.

1. – Introduction.

The purpose of this paper is to verify the validity of a formula [1] given a few years ago to determine the primary energy of jets. This formula is widely used today in the study of interactions at ultrarelativistic energies; it offers some advantage with respect to other formulae, since it makes possible the statistical use of all the data obtainable from the prong angular distribution, and the estimation of the error.

An accurate verification was naturally not possible while the big accelerators were not able to produce energies greater than 1 GeV. Some verifica-

[1] C. Castagnoli, G. Cortini, C. Franzinetti, A. Manfredini and D. Moreno: *Nuovo Cimento*, **10**, 1539 (1953).

tions of the formula have been carried out on 6.3 GeV protons ([2-4]), with not very satisfactory results. As soon as the CERN proton-synchrotron was in operation, we thought that one of the first problems to deal with in the analysis of the nucleon-nucleon interaction (which we are carrying out in collaboration with other laboratories) was the verification of the formula at 27 GeV.

2. – Method of determination of the energy.

The method of determination of the primary energy in p-p collisions is based on the use of the formula given in ([1]), as

$$\lg \gamma = \frac{1}{n_s} \sum_{i=1}^{n_s} \lg |\operatorname{ctg} \theta_i| + \bar{\bar{u}} \pm \frac{\sigma}{\sqrt{n_s}}, \tag{1}$$

where θ_i are the measured angles of the n_s shower particles of the jet, produced by a primary of energy $E_p = M_p c^2 (2\gamma^2 - 1)$; u may be calculated from

$$\bar{\bar{u}} = \int_{-1}^{+1} \int_{0}^{\infty} S(\varepsilon^*, \mu)\, u(\varepsilon^*, \mu)\, \mathrm{d}\varepsilon^* \mathrm{d}\mu\,, \qquad \mu = \cos\theta^*, \tag{2}$$

where $S(\varepsilon^*, \mu)$ is the angular and energy distribution, in the center-of-mass system, of the mesons, deducible from the various theories on meson production, while $u(\varepsilon^*, \mu)$ is given by

$$u = \lg \frac{(1-\mu^2)^{\frac{1}{2}}}{|z+\mu|}\,; \qquad z = \frac{\beta}{\beta^*}\,; \qquad \varepsilon^* = (1-\beta^{*2})^{-\frac{1}{2}}. \tag{3}$$

Formula (1) is based on three hypotheses:

a) that it be a nucleon-nucleon collision;

b) that the angular distribution of the n_s particles of the jet be symmetrical with respect to the equatorial plane in the center-of-mass system;

c) that the n_s particles be independent, that is to say, that there is no correlation between their angles nor between their energies of emission. This hypothesis is justified in spite of the conservation theorems, whenever n_s

([2]) U. HABER-SCHAIM: *Nuovo Cimento*, **4**, 669 (1956).

([3]) H. WINZELER, R. KLAIBER, W. KOCH, M. NIKOLIĆ and M. SCHNEEBERGER: *Nuovo Cimento*, **17**, 8 (1960).

([4]) P. L. JAIN and H. C. GLAHE: *Phys. Rev.*, **116**, 458 (1959).

is large and a considerable number of neutral particles is emitted together with the charged ones observed.

If the further hypothesis is made that $z = 1$ (that is, that $\beta_1^* = \beta$, or that the velocity in the center-of-mass system of the outgoing particles is the same for all and equal to the velocity of the center of mass itself), one says that the formula is, in the first approximation, spectrum-independent. In first approximation, then, for each jet,

$$\lg \gamma_1 = \frac{1}{n_s} \sum_{i=1}^{n_s} \lg |\operatorname{ctg} \theta_i| \pm \frac{\sigma_1}{\sqrt{n_s}} . \tag{4}$$

3. – Experimental details, and results.

3·1. – The so called « Bern beam » ([5]) has been used, that is, a proton beam emitted from the CERN proton-synchrotron working nominally at 28 GeV. The emulsions were 600 μm G-5.

It was assumed that all primary tracks making an angle less than 10′ with the mean direction of the beam where due to protons of the said energy. The momentum spectrum of this beam has been studied ([6]) for circulating protons of 24 GeV, and found to be not monochromatic. Assuming the same momentum distribution for the nominal 27 GeV beam, in our stack the mean energy should be $E_p = 27$ GeV corresponding to $\gamma = 3.92$.

Scanning has been made both by the track following method (Rome) and by track and area scanning (Parma). The star features are very similar so that no sensible bias has been made in the area scanning.

The interaction mean free path ($\lambda = 38$ cm) obtained from track-following indicates a good efficiency for this method; furthermore there seems to be no bias favouring high or low multiplicities n_s.

The stars have been subdivided by us into following categories:

$N_h = 0, 1$: these stars are usually attributed to collisions against nucleons. But since it is reasonable to imagine that some fraction of the stars with $N_h = 1$ is due to collisions against nuclei, we have subdivided this category into two subgroups, with $N_h = 0$ and $N_h = 1$ (with no visible electron or recoil).

$2 \leqslant N_h \leqslant 4$: these stars are for the most part attributed to collisions against light nuclei (C, N, O) ([7]).

$N_h \geqslant 7$: these stars are attributed to collisions against Ag and Br.

([5]) B. Dayton, W. Koch, M. Nicolić, H. Winzeler, J. Combe, W. M. Gibson, W. O. Lock, M. Schneeberger and G. Vanderhaege: *Helv. Phys. Acta*, **33**, 544 (1960).

([6]) A. Wetherell, A. Diddens and G. Cocconi: private communication (7-9-1960).

([7]) M. Friedländer: *Nuovo Cimento*, **14**, 796 (1959).

In the present elaborations all star with $N_h = 0$ found by track and area scanning have been used; while an unbiassed sample has been taken from the $N_h \geqslant 2$ stars so that the statistical weight of the four groups is similar.

Feature of the used sample is in Table I.

TABLE I.

N_h	0	1	2, 3, 4	$\geqslant 7$
Number of stars	74	79	118	100
Forward minimum prongs	464	531	775	900
Total minimum prongs	472	544	795	950

3·2. – In the stars thus classified, the spatial angles θ of all tracks with ionization $g \leqslant 1.3\, g_0$ have been measured. For each jet having $n_s \geqslant 4$ the two values

$$\xi_{1,2} = \frac{1}{n_s} \sum_{i=1}^{n_s} \lg |\operatorname{etg} \theta_i| , \tag{5}$$

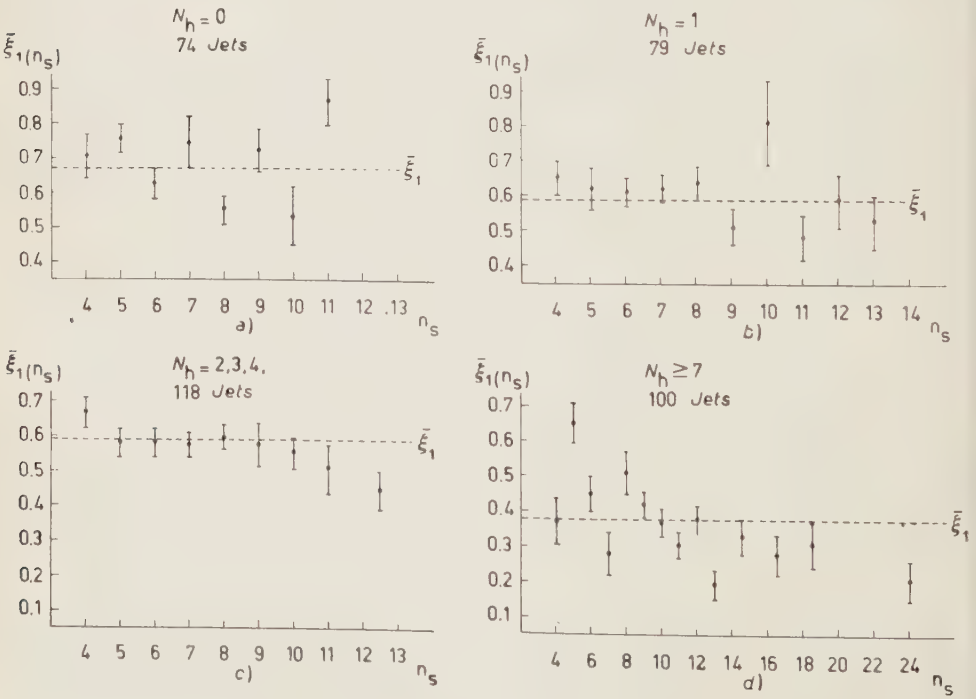

Fig. 1.

1802

have been measured, where for ξ_1, the sum extends over all prongs with $\theta < 90°$, while for ξ_2 it extends to all prongs.

Fig. 1 shows the various values of $\bar{\xi}_1(n_s)$ obtained by averaging the values of ξ_1 for stars with multiplicity n_s. It is seen in Fig. 1*a* that for odd n_s the values of ξ_1 are systematically larger than for even n_s. This is understandable since for n-p collisions hypothesis *b*) of the method is not necessarily valid.

Fig. 1(*a*, *b*, *c*) shows that there is no significant dependence of ξ_1 from n_s for $n_s \geqslant 4$ for $N_h = 0, 1, 2, 3, 4$; for $N_h \geqslant 7$ Fig. 1*d* shows a dependence of ξ_1, from n_s and the average value of ξ_1, $\bar{\xi}_1$, is quite different from the theoretical value, as is to be expected since for this category of jets the number of successive collisions increases.

We neglected multiplicities $n_s \leqslant 4$ on the following grounds:

1) hypothesis *c*) of Section **2** is valid only for n_s sufficiently large;

2) it is furthermore evident that for low multiplicities the influence of the outgoing proton (or protons) on the value of ξ_1 is stronger than at high multiplicities;

3) for the subgroup $N_h = 0$ it is evident that one cannot include the stars with $n_s = 1, 2$ while for those with $n_s = 3$, a contamination of tridents is possible. For this reason a fair comparison of the four groups of stars defined in Section **3**·1 is possible only for $n_s \geqslant 4$.

Averaging all $\xi_{1,2}$ corresponding to these stars, we obtain

$$\bar{\xi}_{1,2} = \frac{\sum \xi_i/\mu_i^2}{\sum 1/\mu_i^2}. \tag{6}$$

The weight $1/\mu_i^2$ is given by $[\sigma_{1,2}^2/(n_s)_i]^{-1}$.

The various values of $\bar{\xi}_1$ and $\bar{\xi}_2$ for the various types of stars, as functions of N_h, are given in Table II.

TABLE II.

	$\bar{\xi}_1$	γ_1	E_1	$\bar{\xi}_2$ exp.	$\bar{\xi}_2$ corr.	γ_2	E_2
$N_h = 0$	0.67±.02	4.7±.1	40±2	0.66	0.61±.02	4.1±.1	31±2
$N_h = 1$	0.59±.02	3.9±.1	28±2	0.58	0.53±.02	3.4±.1	21±1
$N_h = 0, 1$	0.63±.02	4.3±.1	33±1	0.62	0.57±.02	3.7±.1	25±1
$2 \leqslant N_h \leqslant 4$	0.59±.02	3.9±.1	27±2	0.52	0.47±.02	3.0±.1	16±1
$N_h \geqslant 7$	0.38±.02	2.4±.1	10±1	0.32	0.27±0.2	1.9±.1	6±1

The quoted errors are the mean square deviations of the experimental distributions of $\lg|\operatorname{ctg}\theta_i|$.

4. – Energy determination.

4·1. *First approximation.* – If the distribution of ξ_1 is normal, we obtain from formula (4) and from (5)

$$\gamma_1 = 10^{(\bar{\xi}_1 + (\sigma_1^2/2))}$$

and the variance

$$\sigma_{\gamma_1}^2 = \gamma_1^2 (10^{\sigma_1^2} - 1) .$$

The measured γ_1 and the corresponding E_1 (to be compared with the expected value of $E = 27$ GeV) are shown in Table II.

The agreement is quite satisfactory for the first three groups of stars (*i.e.* $N_h = 0, 1, ..., 4$). For stars with $N_h \geqslant 7$ the disagreement is obviously attributable to the existence of secondary collisions within the nucleus, as was to be expected.

4·2. *Second approximation.* – The first approximation is, however, less satisfactory at the relatively low energy at which we are verifying the formula, than at the very high energies of the cosmic radiation, which is the field where it is normallly used. We have thus tried to verify the behaviour of the formula in the second approximation.

The calculation of u has been carried out under the assumption of Fermi's theory. At our energy, in Fermi's termodynamic approximation and under the hypothesis of center-of-mass isotropy, is (*)

$$\bar{u} = -0.05 .$$

Considering the pion spectrum calculated at 25 GeV by HAGEDORN (8) in the case of the statistical model numerical calculation gives $u = -0.06$.

It may be assumed, concluding, that at 27 GeV $\bar{u} = -0.05$. This value is in excellent agreement with the experimental value found at comparable energies by JAIN *et al.* (9).

In this second approximation it is necessary to use the value $\bar{\xi}_2$ instead of the mean experimental value $\bar{\xi}_1$. One obtains, the values of γ_2 and E_2 given in Table II.

(*) Calculations are in Appendix A.

(8) R. HAGEDORN: *Nuovo Cimento*, **15**, 434 (1960).

(9) P. L. JAIN, E. LOHRMANN and M. V. TEUCHER: *Phys. Rev.*, **115**, 643 (1959).

4·3. *Dispersion of the measurements.* – It has been shown [1] that the $\log|\operatorname{cotg}\theta|$ distribution is normal with a dispersion depending on the angular and energy distributions in the c.m.s. Appendix B deals with this point in greater detail. It is found in the first and second approximations, respectively, and with isotropic distribution in the c.m.s.,

$$\sigma_{0,1} = 0.39\,, \qquad \sigma_{0,2} = 0.37\,.$$

Our $\lg|\operatorname{ctg}\theta|$ distributions are normal, with the present statistics, and our values of $\sigma_{1,2}$ are

$$\sigma_1 = 0.50\,, \qquad \sigma_2 = 0.52\,,$$

for all star groups.

Bearing in mind that (B.1) gives for $K=1$ $\sigma_{1,1}=0.59$ this experimental value of $\sigma_{1,2}$ can be an indication of a departure from isotropy in c.m.s. But the result may also be due to successive collisions, expecially in stars with a large number of heavy prongs, or, for $N_h = 0, 1$, to the characteristics of the distribution of ξ connected with the physics of the p-nucleon collisions. We don't go into this in the present work.

Of course, the verification of anisotropy in the center-of-mass system may be carried out directly, comparing the angular distribution in the center of mass with the theoretical distribution. Note, however, that, in the second approximation, this comparison requires knowledge of β^* for the single particles.

5. – Discussion and conclusions.

As was said in the introduction, formula (1) has been already tested experimentally at lower energies. A disagreement has been found which could be summarized, essentially, in the following facts:

1*a*) the E_0 thus determined was strongly dependent on n_s;

2*a*) even within the region of n_s where the formula gives the best approach, the primary energy is overestimated by a factor between 0.3 and 2 [3];

3*a*) the energy determinations are subject to very large fluctuations, n_s remaining constant.

From our experimental results we can draw the following conclusions:

1) for jets with $N_h = 0, 1$ formula (1) gives in the second approximation a result in very good agreement with the expected one;

2) for $N_h = 2, 3, 4$ agreement is good in the first approximation but not in the second. This is justified by the fact that successive collisions tend to increase the number of backward prongs (cf. Table II) and hence in this case the experimental ξ_2, has a different meaning than that required by the second approximation;

3) for jets with $N_h \geqslant 7$, for which the hypotheses of formula (1) are not correct the disagreement between experimental and expected value is noticeable also in the first approximation, the energy is underestimated, as one would expect since the considered jets are due to interactions with nuclei;

4) from our experimental results (Fig. 1*a*, *b*, *c*) one can say, that, for stars with $n_s \geqslant 4$, the dependence of the calculated ξ on n_s is rather small (whatever the mean value).

All this suggests that use of formula (1) gives statistically good results only for jets with $n_s \geqslant 4$ and $N_h \leqslant 4$, that is, for those interactions which are usually defined as jets in cosmic ray studies.

The disagreement between our results and other results obtained at lower energies, can be probably imputed to the use of stars with low moltiplicity n_s and to the fact that at lower energies than 27 GeV, the first approximation of formula (1) is probably no more reliable and second approximation should be used.

* * *

We are very indebted to the CERN emulsion group for providing the exposure of the stack.

Appendix A

Assuming in the (2) that $S(\varepsilon^*, \mu) = S(\varepsilon^*) f(\mu)$ one can compute the integral first with respect to μ; assuming isotropy in the c.m., one obtains

$$\bar{u}(z) = \int_{-1}^{+1} u(\varepsilon^*, \mu) f(\mu)\, \mathrm{d}\mu = \lg 2 - \frac{1}{2} z \lg \left| \frac{z+1}{z-1} \right| - \frac{1}{2} \lg |z^2 - 1|, \tag{A.1}$$

with

$$= \beta(\gamma') \Big|_{\varepsilon^{*2}}^{\varepsilon^{*2}} 1 \cdot$$

This function may be represented in the region of z of interest, given our energy, as a straight line:

$$u'(z) = -\,0.52(z-1)\,. \tag{A.2}$$

Integration with respect to energy may be carried out analitically, now

$$\bar{\bar{u}} = \int_0^\infty S(\varepsilon^*)\bar{u}'(z)\,\mathrm{d}\varepsilon^* = -\,0.52\int_0^\infty z\,S(\varepsilon^*)\,\mathrm{d}\varepsilon^* + 0.52\int_0^\infty S(\varepsilon^*)\,\mathrm{d}\varepsilon^*\,.$$

Using Fermi's theory, a first form of the spectrum may be obtained in the thermodynamic approximation already dealt with by us [1]. We obtain (*)

$$\bar{\bar{u}} - 0.52 = -\,0.52\,\frac{\beta(\gamma)}{B}\int_0^\infty \sqrt{1+(Bx)^2}\,\frac{x\,\mathrm{d}x}{e^x-1} = \tag{A.4}$$

$$= -\,0.52\beta(\gamma)\left\{\left[\frac{1}{B}\,G_1\left(\frac{1}{B}\right) + B^2 G_3\left(\frac{1}{B}\right)\right] + \left|G_2(\infty) - G_2\left(\frac{1}{B}\right)\right| + \right.$$

$$\left. + \frac{1}{2B^2}\left[\frac{1}{B} - \ln\left(\exp\left[\frac{1}{B}-1\right]\right)\right]\right\},$$

where

$$G_k(x) = \int_0^\infty \frac{x^k}{e^x-1}\,\mathrm{d}x\,.$$

At our energy is

$$\bar{u} = -\,0.05\,.$$

Appendix B

We consider here the form and the dispersion of the distribution of the single values of lg ctg θ_1. Starting from the fundamental relation (valid in the first approximation)

$$v_1 = \gamma\,\mathrm{tg}\,\theta = \frac{\sin\theta^*}{1+\cos\theta^*} = \mathrm{tg}\,\frac{\theta^*}{2}\,,$$

(*) To be compared with (B.3) of paper [1], mistaken by an error of approximation.

and assuming the distribution in the c.m. system to be isotropic, one obtains, for the distribution function $f(u_1)$ of the quantity $u_1 = \lg r_1 = \lg \gamma \operatorname{tg} \theta$ in the laboratory system

$$f(u_1) = \frac{1}{\cosh^2 u_1} .$$

We have shown in (1) that $f(u_1)$ may be approximated by a gaussian function:

$$\varphi(u_1) = \frac{1}{\sqrt{2\pi}\sigma_{01}} \exp\left[-\frac{u_1^2}{2\sigma_{01}^2}\right] ,$$

with

$$\sigma_{01}^2 = \frac{\pi^2}{12} (\lg_{10} e)^2 .$$

If, in place of the isotropic distribution $K = 0$ one considers a symmetrical distribution represented by $(K-1)/2 \cos^{2K} \theta^*$, one must use, instead of σ_{01} a value σ_{K1} given by

$$\sigma_{K1}^2 = \left(\frac{\pi^2}{12} + \sum_{i=1}^{K} \frac{1}{i} \sum_{s=0}^{i-1} \frac{1}{2s+1}\right) (\lg_{10} e)^2 .$$

If one whishes to calculate the effect of the second approximation on σ it is necessary to calculate the expression

$$\sigma_{K2}^2 = \frac{2K+1}{2} \int_0^\infty S(\varepsilon^*)\, \mathrm{d}\varepsilon^* \int_{-1}^{+1} \mu^{2K} \lg^2 \frac{(1-\mu^2)^{\frac{1}{2}}}{|z+\mu|}\, \mathrm{d}\mu = \int \overline{u_K^2(z)}\, S(\varepsilon^*)\, \mathrm{d}\varepsilon^* .$$

For the calculation of σ_{K2}, it is necessary to make the following integral:

$$u_K^2(z) = \frac{2K+1}{2} \int_{-1}^{+1} \mu^{2K} \lg^2 \left(\frac{1}{|z+\mu^2|}\right) \mathrm{d}\mu .$$

We report here this integral as given to us by Dr. W. Gross:

$$\begin{aligned} u_K^2(z) = &- f\left(\left|\begin{matrix} z & 1 \\ z & -1 \end{matrix}\right|\right) + \frac{1}{4} \lg^2 (z^2-1) - \frac{1}{4} z^{2K+1} \lg^2 \frac{z+1}{z-1} - \lg 2 \lg (z^2-1) + \\ &- \left[\sum_{i=1}^{K-1} \frac{z^{2K-1}}{2i+1} - \sum_{i=0}^{K-1} \frac{1}{2i+1}\right] \lg (z^2-1) + \\ &- \left[z^{2K+1} \sum_{i=1}^{2K} \frac{1}{i} - \sum_{i=1}^{K} \left(\frac{1}{2i} + \frac{1}{2(K-i)+1}\right) z^{2(K-i)+1}\right] \lg \frac{z+1}{z-1} + \\ &+ 2 \sum_{i=1}^{K-1} a_i^{(K)} \frac{z^{2(K-i)}}{2i+1} + C_K , \end{aligned}$$

where

$$f(\xi) = -\int_0^{\xi} \frac{\lg(1-x)}{x}\,\mathrm{d}x = \sum_{i=1}^{\infty} \frac{\xi^i}{i^2},$$

$$a_i^{(K)} = \log 2 + \sum_{2i+1}^{2K+1} \frac{1}{s} + \frac{2i+1}{2(K-i)} \sum_{K-i}^{K} \frac{1}{2s+1} - \frac{2K+1}{2(K-i)} \sum_{s=0}^{i} \frac{1}{2s+1},$$

$$C_K = \left(\sum_0^K \frac{1}{2i+1} - \lg 2 \right)^2 - \sum_0^K \frac{1}{(2i+1)^2} - \frac{2}{2K+1} \sum_0^{K-1} \frac{1}{2i+1} + \frac{2\lg 2}{2K+1} + \frac{\pi^2}{12}.$$

In the particular case of $K = 0$ one obtains

$$u_0^2(z) = -\sum_{i=1}^{\infty} \left| \frac{z-1}{z+1} \right|^i \frac{1}{i^2} + \frac{1}{4} \lg^2 (z^2-1) - \frac{1}{4} z \lg^2 \left| \frac{z+1}{z-1} \right| +$$

$$- \lg 2 \lg (z^2-1) + (\lg 2)^2 + \frac{\pi^2}{12}.$$

RIASSUNTO

Sono state analizzate le interazioni prodotte da protoni di 27 GeV in emulsione nucleare allo scopo di controllare a questa energia la validità della formula $\lg \gamma_c = \langle \lg |\mathrm{ctg}\,\theta| \rangle$. È stato trovato un buon accordo fra il valore previsto ed il valore calcolato per le interazioni con $N_h \leqslant 4$ e $n_s \geqslant 4$. Si discute la correzione da introdurre nell'ipotesi che lo spettro dei mesoni prodotti sia quello previsto dalla teoria statistica, e la dispersione dei valori di $\lg \gamma_c$ ottenuti.

Latitude-Dependence of a Forbush Type Decrease.

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(ricevuto il 27 Gennaio 1961)

Summary. — Latitude-dependence of a cosmic ray intensity decrease has been found. It has been shown that the relative amplitude does not flatten beyond the latitude knee. It, rather, goes on increasing even much beyond the knee. It is also seen that the relative amplitude is not symmetrical around the geomagnetic equator.

The decrease in cosmic ray intensity during a magnetic storm was observed as early as 1933 by Messerschmidt [1] and by Steinmaurer and Graziadei [2]. Later, Forbush [3] using the data of widely separated stations—Huancayo, Cheltenham and Hafelekar—showed for the first time that the decrease in cosmic ray intensity with magnetic storm is of world-wide character. Since then many reports of decreases in cosmic ray intensity with magnetic storms have appeared in literature and it has been shown that all the magnetic storms do not produce cosmic ray intensity decreases [4].

Various arbitrarily chosen methods for measuring the amplitude of the decreases have been used by different workers. The essential idea is to take the difference between the lowest intensity recorded and the normal intensity. For normal intensity some workers have taken the intensity at the point where the sharp decrease starts while the others take it as the average of that of a few days preceding the commencement of the storm. Fenton *et al.* [5] used

[1] W. Messerschmidt: *Zeits. Phys.*, **85**, 332 (1933).

[2] R. Steinmaurer and H. T. Graziadei: *Sitz. Preuss. Akad. Wiss. Berlin*, **22**, 672 (1933).

[3] S. E. Forbush: *Phys. Rev.*, **51**, 1108 (1937).

[4] H. Elliot: *Progress in Cosmic Ray Physics*, vol. **1** (Amsterdam, 1952).

[5] A. G. Fenton, K. B. Fenton and D. C. Rose: *Can. Journ. Phys.*, **36**, 824 (1958).

another quantity—average departure—defined as the area of the decrease *i.e.*, the product of decrease and time. Later FENTON *et al.* [6] evolved a different technique for determining the depth of the Forbush decrease.

MCCRACKEN and JOHNS [7] suggested a still different method. According to them if the data, R_1 and R_2, of two stations show a correlation coefficient greater than 0.8, then a quantity named « relative amplitude » gives a good measure of the amplitude of the decrease. The relative amplitude is defined as

$$(\sigma_1/\bar{R}_1)\big/(\sigma_2/\bar{R}_2)\,,$$

where

$$\sigma_i = \sqrt{[1/(N-1)]\sum(R_i-\bar{R}_i)^2}\,,$$

$$\bar{R}_i = [1/N]\sum R_i$$

and R_i is the daily mean intensity.

MCCRACKEN [8] has discussed the relative advantages of using this measure. In the present analysis we have used the method of McCracken and Johns [7].

Using the definition $(\sigma_1/R_1)\big/(\sigma_2/R_2)$, the relative amplitudes of the nucleonic intensity for ten sea-level stations with respect to Huancayo have been computed for the period Aug. 24 to Sept. 20, 1957. During this period, there occured a magnetic storm accompanied by a cosmic ray storm. The pressure corrected neutron monitor data, supplied by the National Committee for the

TABLE I. – *Relative amplitude of ten sea-level stations.*

Station	Geomagnetic latitude	Correlation coefficient	Relative amplitude
Ahmedabad	13.9	0.90	0.94
Lae	— 16	0.95	0.74
Makapuu Point	21.3	0.69	1.39
Buenos Aires	— 23.3	0.95	1.01
Hermanus	— 33.7	0.87	1.33
Berkeley	44.1	0.97	1.73
Herstmonceux	53.5	0.97	1.92
Uppsala	58.6	~0.8	2.04
Mawson	— 73.1	~0.8	2.07
Resolute	82.9	0.97	2.17

(6) A. G. FENTON, K. G. MCCRACKEN, D. C. ROSE and B. G. WILSON: *Can. Journ. Phys.*, **37**, 569 (1959).

(7) K. G. MCCRACKEN and D. H. JOHNS: *Nuovo Cimento*, **13**, 96 (1959).

(8) K. G. MCCRACKEN: *Phys. Rev.*, **117**, 1570 (1960).

I.G.Y., Science Council of Japan, have been used. A preliminary report of this analysis has appeared elsewhere (9). The results are presented in Table I and have been plotted in Fig. 1.

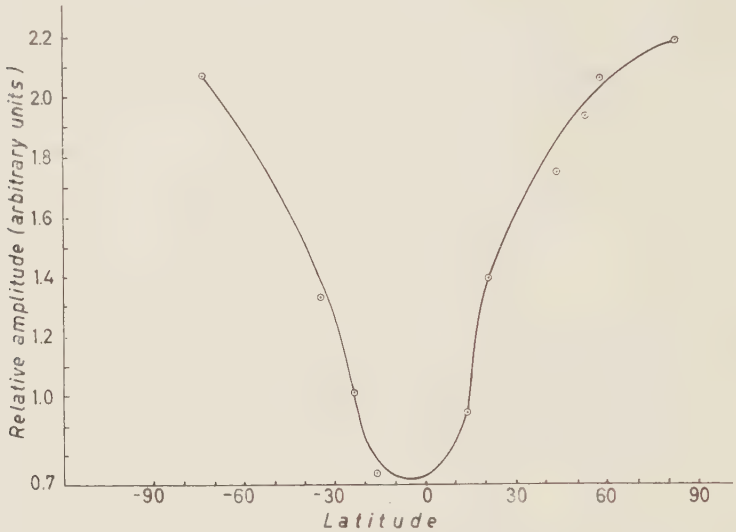

Fig. 1. – Latitude-dependence of a Forbush-type decrease. The relative amplitude is in arbitrary units.

We see from Fig. 1 that the relative amplitude goes on increasing with latitude on both sides of the geomagnetic equator. LOCKWOOD (10), while studying the latitude-dependence of a cosmic ray intensity decrease (on Feb. 11, 1958), found that the amplitude of decrease goes on increasing up to around the latitude knee stations but then remains almost constant. However, in the present case the amplitude is seen to be increasing up to 82.9° geomagnetic latitude in the northern hemisphere and 73° geomagnetic latitude in the southern hemisphere. The data beyond these points, unfortunately, are not available. SINGER (11) has given two latitude *vs.* intensity curves for neutron data taken in aircraft. Data for one of them were taken during the magnetic storm period (June 18 to June 23, 1951) while the other represents the data taken immediately after this storm. The trends of the two curves suggest that the amplitude *vs.* latitude curve, deduced from them, will get flattened beyond the latitude knee. The discrepancy between the present case and the

(9) LEKH VIR and P. S. GILL: *Ind. Journ. Phys.*, **34**, 531 (1960).
(10) J. A. LOCKWOOD: *Journ. Geophys. Res.*, **65**, 27 (1960).
(11) S. F. SINGER: *Progress in Elementary Particles and Cosmic Ray Physics*, vol. **4** (Asterdam, 1958).

others may be due to the following reasons: 1) The mechanism responsible for causing this particular cosmic ray decrease may be different from that for others. However, this does not seem probable because the other features of the decrease do not differ from the normal ones. 2) The method of analysis may be responsible for it. As has been pointed out by McCracken [8], this method is less sensitive to changing asymmetry and to the residual meteorological contributions, the results obtained this way represent therefore the general trend better.

Further, it is obvious from the curve that the entire range of the energy spectrum, covered by this analysis, has been effected. The effect, however is not uniform over the entire range. It is maximum on the lowest part of the spectrum and goes on decreasing with increasing energy. Also we see that the relative amplitude is not symmetrical about the geomagnetic equator.

McCracken and Johns [7] have found that for stations, situated beyond the latitude knee, the relative amplitude varies by about 12% per thousand metres increase in altitude. In their analysis, they have tacitly assumed that the relative amplitude-latitude curve is symmetrical about the geomagnetic equator and flattens beyond the latitude knee. In the light of the present work, however, this assumption does not appear to be valid. Hence there is a necessity for revaluating the change in relative amplitude with altitude.

* * *

We are thankful to Mr. M. K. Khera for the help in preparing the manuscript and to the National Committee for the I.G.Y., Science Council of Japan for supplying the pressure corrected neutron monitor data of the stations participating in the I.G.Y. programme.

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RIASSUNTO (*)

Abbiamo trovato la dipendenza dalla latitudine di una diminuizione della intensità dei raggi cosmici. Abbiamo mostrato che l'ampiezza relativa non si appiattisce dopo il gomito della curva in funzione della latitutine. Anzi continua a crescere ancora molto dopo tale gomito. Facciamo anche vedere che l'ampiezza relativa non è simmetrica rispetto all'equatore geomagnetico.

(*) *Traduzione a cura della Redazione.*

Properties of a Spark Chamber (*).

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(ricevuto il 6 Febbraio 1961)

Summary. — A spark chamber has been constructed of seven aluminum plates, each $\frac{1}{2}$ in. thick and separated by a $\frac{1}{4}$ in. gap. Using this device permits the trajectory of a charged particle to be observed by detecting the light produced by a spark discharge between the plates when a high-voltage pulse is applied to alternate plates. The efficiency of this chamber has been measured for high-energy charged particles produced at the Bevatron. When the chamber is filled with 1 atm of neon or argon, the efficiency per gap is greater than 99% if a $\frac{1}{4}$ μs pulse of greater than 10 kV for neon or 15 kV for argon is applied to the plates within $\frac{1}{3}$ μs after the charged particle traverses the gap between the plates. Multiple tracks, such as those produced by a nuclear reaction in the plates can easily be photographed. The recovery time of the chamber, after a spark discharge between the plates, is approximately 10 ms. The sensitive time is of the order of 10 μs unless a clearing field is applied to one set of the plates. With a clearing field of approximately 160 V/cm, the sensitive time is reduced to less than $\frac{1}{2}$ μs. When the chamber had $4 \cdot 10^5$ particles incident on it in 0.1 s, three tracks per trigger were sometimes observed, in agreement with the measured sensitive time. The chamber also functioned with a magnetic field of 13 kG parallel to the plates. The tracks were displaced by an amount proportional to the magnetic field. In addition, if a high clearing field, E, was applied and if a long time was allowed to elapse before the high-voltage pulse was applied, the sparks in alternate gaps were displaced in opposite directions by an amount proportional to $\boldsymbol{E} \times \boldsymbol{B}$. Both the momentum and the time of traversal, relative to the time of applied voltage, could be determined in this manner.

(*) This work was done under the auspices of the U.S. Atomic Energy Commission.

(**) On leave from the National Institute for Research in Nuclear Science, Harwell England.

Certain kinds of experiments in high-energy physics could benefit from the use of a charged-particle detector with high spatial resolution combined with short time resolution. At present, the former characteristic is found in bubble chambers and photographic emulsions; the latter in scintillation and Čerenkov counters. A discharge chamber that has both of these properties has been described by Cranshaw and De Beer [1] and also by Fukui and Miyamoto [2]. This paper describes a similar device which has been tested in a beam of high-energy particles from the Bevatron.

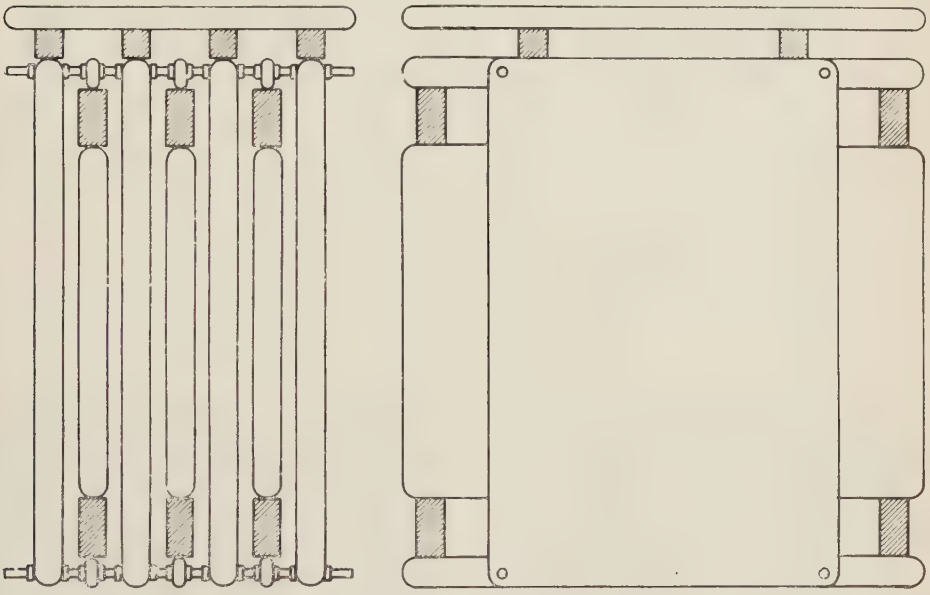


Fig. 1. – Diagram of the spark chamber.

The spark chamber is shown in Fig. 1; a block diagram of the electronics is shown in Fig. 2. A charged particle that passes through counter telescope $C_1C_2C_3$ also passes through the spark chamber in a direction approximately normal to the plates. The results reported here have been obtained with a chamber containing argon; the sensitive area was 6 by 6 in. A minimum-ionizing particle produces about 30 ion pairs in neon in each $\frac{1}{4}$-in. gap. A coincidence in the counter telescope triggers an EFP60 discriminator circuit, which in turn triggers a 6130 hydrogen thyratron. This applies a 2 kV pulse to a 5949 thyratron, which in turn applies a pulse of up to 28 kV to alternate

(1) T. E. Cranshaw and J. F. De Beer: *Nuovo Cimento*, **5**, 1107 (1957).
(2) S. Fukui and S. Miyamoto: *Nuovo Cimento*, **10**, 113 (1959).

plates of the chamber. The rise time of this latter pulse is 25 ns. Within 0.1 μs of the application of this pulse, a spark discharge occurs in each gap

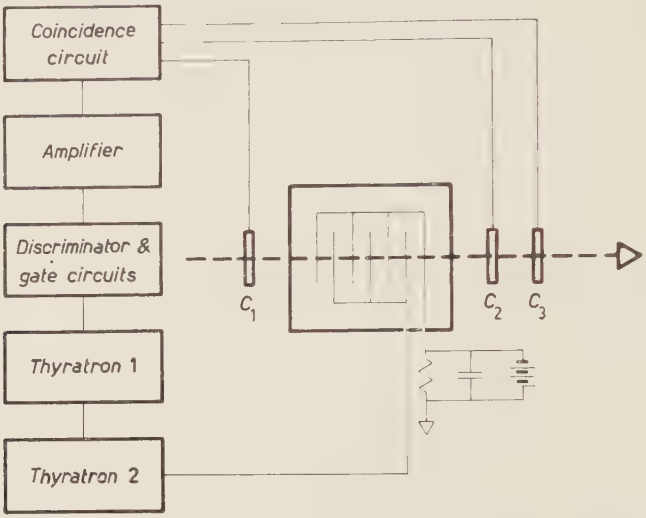
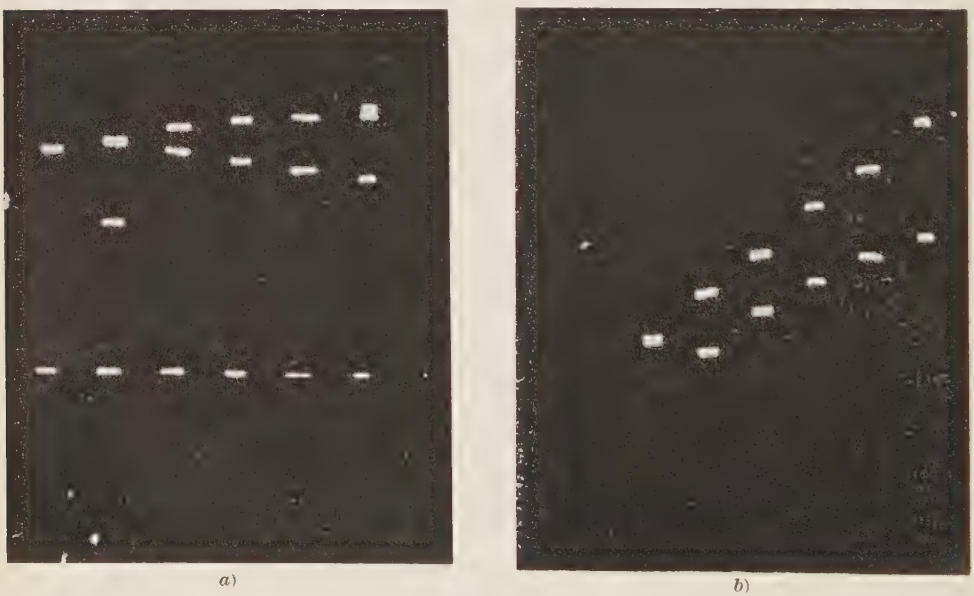


Fig. 2. – Block diagram of electronics. Hydrogen thyratrons 1 and 2 are Kuthe type 6130 and 5949, respectively.



a) *b*)

Fig. 3. – Typical photographs obtained with the spark chamber. *a*) Two particles passing through the chamber. One has interacted in a plate. Minimum ionizing particles enter from the left. *b*) Two particles passing through the chamber obliquely, one at 36°, the other at 25° to the normal to the plates.

at the point where the particle passed. With the help of a mirror, 90 stereo photographs of the discharge were recorded on Tri-X film in a single camera. The aperture was $f/11$ with the camera 5 ft. away. Some typical pictures are reproduced in Fig. 3. In order to obtain good time resolution, a constant clearing field was maintained between the plates; electrons from the traversal of off-time particles were thereby swept out.

The time resolution was measured by delaying the application of the high voltage to the plates. The minimum time between the passage of the particle and the arrival of the high-voltage pulse at the chamber was 0.3 μs; most of this was due to the triggering delay of the thyratrons. Fig. 4 shows the variation of efficiency of a single gap in argon as a function of additional delay for various clearing fields. The curves show that it is possible to achieve a resolution time of 0.5 μs with an efficiency better than 99%. The efficiency for sparks to appear in all six gaps was always approximately the sixth power of the efficiency of a single gap, showing that the gaps behaved independently. Similar curves are obtained when the chamber is filled with neon or helium.

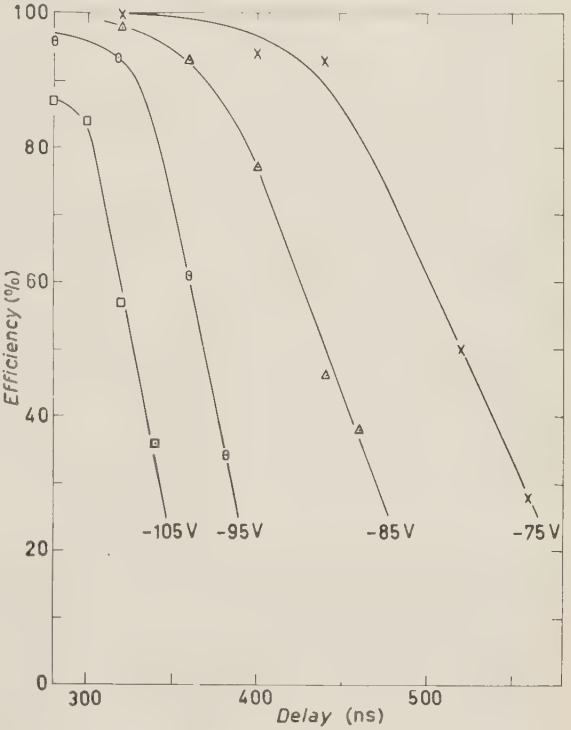


Fig. 4. – Efficiency of a single $\frac{1}{4}$ in. gap in 1 atm of argon as a function of delay in application of the high-voltage pulse. The zero of the delay axis is the time at which the particle passed through the chamber.

The curves of Fig. 4 were obtained with the clearing field in the opposite direction to the pulsed field. With the clearing field in this direction, the electrons—which initiate the discharge—are swept towards one plate. There is then sufficient room for development of spark breakdown in an electron avalanche moving towards the other plate. With a clearing field of the same magnitude in the other direction, high efficiency was not possible because of the non-zero distance required for the development of a spark [3]. This dis-

[3] J. M. Meek and J. D. Craggs: *Electrical Breakdown in Gases* (Oxford, 1953).

tance decreases with increasing amplitude of the high-voltage pulse; when the voltage is low enough for the breakdown distance to be comparable with the gap width, the efficiency is reduced. This effect is shown in Fig. 5.

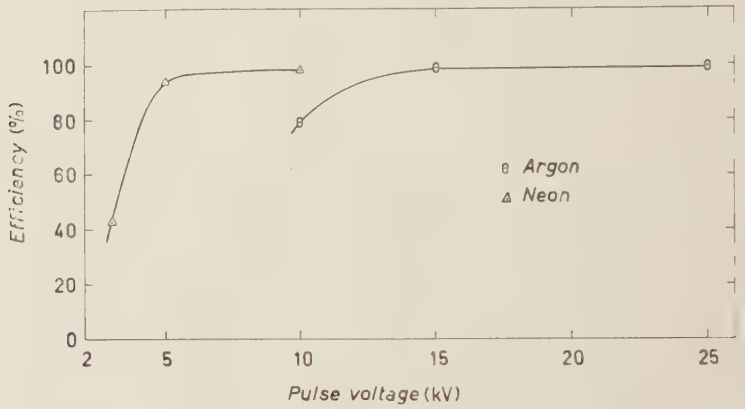


Fig. 5. – Efficiency of a single $\frac{1}{4}$ in. gap in 1 atm of neon and argon as a function of pulse voltage.

The recovery time of the chamber was measured by triggering the chamber with two particles occurring a fixed time apart. The results are summarized

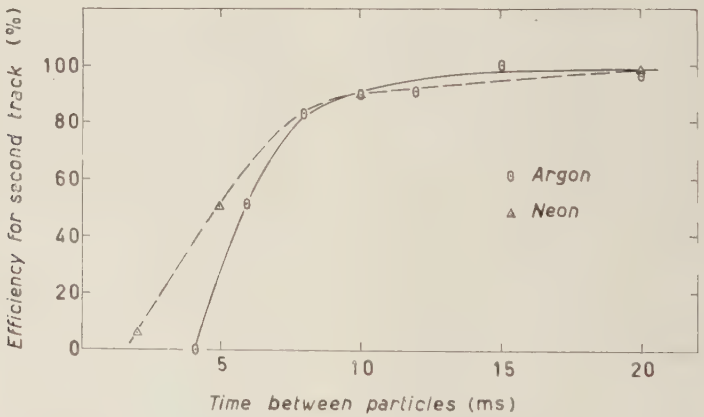


Fig. 6. – Efficiency for a single gap to spark on a second particle as a function of the time between particles. The clearing field was — 40 V/cm.

in Fig. 6. These curves were *not* controlled by the recovery of the high-voltage supply; at 1 ms delay, the high voltage had recovered to a level at which the efficiency is 60%.

An attempt was made to find a clearing field that would give a low efficiency

Fig. 7. – Effect of a magnetic field of 13 kG with a clearing field 80 V/cm and a time delay of 1 μs. The gas was argon.

for particles with minimum ionization and high efficiency for densely ionizing particles. With a clearing field of 47 V and a delay of 0.6 μs in neon, the efficiency was 60% for minimum ionization and 80% for three-times-minimum ionization.

With all parameters held fixed, it was found that the resolving time of the chamber increased during a period of several days after evacuating and refilling the chamber. Most of the change occurred in the first few hours after the chamber was filled. This effect was presumably due to the change in concentration of some impurity. The resolving time was restored to the initial value by an increase in the clearing field. Addition of $\frac{1}{2}$% oxygen or 5% carbon dioxide did not affect the efficiency appreciably. Greater quantities did reduce the efficiency, presumably by reducing the probability that an electron would survive until the high-voltage pulse was applied. These added impurities did not affect the recovery time. Addition of a small amount ($\sim\frac{1}{5}$%) of argon to neon did not have any marked affect on the brightness of the spark, nor on the time resolution. In the $\frac{1}{4}$ in. gap, no tendency was seen for the spark in a single gap to follow the track of the particle for tracks at angles up to 45° to the normal to the plates; with angles greater than this, multiple sparks occurred. One of the pictures in Fig. 3 shows the appearance of such a track. In tests with a $\frac{1}{2}$ in. gap some sparks did partially follow the particles in the same way as shown in the results of FUKUI and MIYAMOTO [2], but the effect did not appear consistently.

The estimate of the resolving time obtained as described above was confirmed by tests in a beam of $4 \cdot 10^6$ particles per second; approximately three tracks per trigger were observed in the chamber. This result, together with observation of interaction events in the plates leading to secondary particles, shows that there is no loss of efficiency when several particles are detected at once.

When the chamber was placed in a magnetic field perpendicular to the particle trajectory, the sparks in alternate gaps were displaced in opposite directions as in Fig. 7. The direction of displacement was $-\boldsymbol{E}\times\boldsymbol{B}$, with $\boldsymbol{E}$ the clearing field. This is due to the familiar cycloidal motion of the electrons in the crossed fields, modified by the presence of the gas. This situation is similar to that used by TOWNSEND in his measurements of ionic mobilities [4]. With a clearing field of 80 V/cm, a delay of 1 μs, and a magnetic field of 13 kG, the relative displacement in successive gaps was about 1 cm. This displacement is proportional to the lapse of time between passage of the particle and application of the high voltage. It is conceivable that this effect could be used to reject accidental particles; an out-of-time track would have an incorrect displacement. The time resolution might be made as short as 0.1 μs

[4] J. TOWNSEND: *Electrons in Gases* (New York, 1947), p. 20.

in this way. The efficiency of the chamber was still greater than 99% in a magnetic field of 17.5 kG.

Gases of ordinary purity were used; welder's argon, for example, was found to be quite satisfactory. For this reason values of electron mobilities that can be deduced from this work should not necessarily agree with published values. It is well known that minute quantities of impurities have drastic effects on mobilities [5].

It is concluded that it is possible to construct a spark chamber which has much better spatial resolution than any practical counter array and a time resolution of less than 0.5 μs. Its use could greatly simplify an experiment in which accurate observation of a rare event in an intense flux of particles is required.

* * *

It is a pleasure to thank Professor JAMES CRONIN of Princeton University for many helpful discussions.

[5] See for example J. C. BOWE: *Phys. Rev.*, **117**, 1411 (1960).

RIASSUNTO (*)

Si è costruita una camera a scintille contenente sette lastre di alluminio, ciascuna avente uno spessore di $\frac{1}{2}$ in. e separate da un intervallo di $\frac{1}{4}$ in. L'uso di questo apparecchio permette di scoprire la traiettoria di una particella carica osservando la luce prodotta da una scarica a scintilla che si genera fra le lastre quando un impulso istantaneo ad alto potenziale è applicato a lastre alterne. Si è misurata l'efficienza di questa camera per le particelle di alta energia prodotte dal Bevatrone. Quando la camera è riempita con neon od argon alla pressione di 1 atm, l'efficienza di ogni intervallo è maggiore del 99% se si applica alle lastre un impulso di $\frac{1}{4}$ μs ad un potenziale maggiore di 10 kV per il neon e 15 kV per l'argon entro $\frac{1}{3}$ μs dopo che la particella carica ha attraversato lo spazio fra le lastre. Tracce multiple, come quelle prodotte da una reazione nucleare entro le lastre, possono venire facilmente fotografate. L'intervallo di ricupero della camera, dopo la produzione di una scintilla fra le lastre, è circa 10 ms. L'intervallo di sensibilità è dell'ordine di 10 μs se non si applica ad un gruppo di lastre un campo di spazzamento. Con un campo di spazzamento di circa 160 V cm l'intervallo di sensibilità si riduce a meno di $\frac{1}{2}$ μs. Quando sulla camera incidevano $4 \cdot 10^5$ particelle in 0.1 s, si sono osservate talvolta tre tracce per scarica, in accordo con l'intervallo di sensibilità misurato. La camera ha anche funzionato con un campo magnetico di 13 kG parallelo alle lastre. Le tracce erano spostate di uno spazio proporzionale al campo magnetico. Poi, se veniva applicato un forte campo di spazzamento, E, e se si lasciava trascorrere un lungo intervallo prima di applicare l'impulso ad alto potenziale, le scintille negli spazi alterni erano spostate in direzioni opposte di una quantità proporzionale ad $\boldsymbol{E} \cdot \boldsymbol{B}$. In questo modo si possono determinare sia il momento sia l'istante dell'attraversamento, relativamente all'istante dell'applicazione del potenziale.

(*) *Traduzione a cura della Redazione.*

ISTITUTO DI FISICA TEORICA - UNIVERSITA' DI NAPOLI

INTERNATIONAL SCHOOL OF PHYSICS

Under the Auspices of the Italian Physical Society and of the University of Naples

COURSE ON BIOPHYSICS

July 10 to 23, 1961 - Naples (Italy)

This Course, which is part of the Advanced Study institute programme of NATO, will be held in Naples in the period from July 10th to 23rd, 1961. It is dedicated to MOLECULAR BIOLOGY, its main scope is that of bringing together leading workers in the field and advanced students, with a view to stimulating informal discussions among all participants; a limited number of lectures and seminars will serve to bring into focus the arguments of interest.

Invitations have been extended to a large number of experts; a list of lecturers will be made known as soon as possible, together with a tentative program. The detailed program will be decided in a general discussion to be held on July 10th.

All lectures, seminars, conferences and general discussions will be held in English or French.

Those wishing to attend the School should send, not later than May 31st, 1961 an application to the Director of the School giving the following information: 1) Christian name and surname; 2) Date and place of birth; 3) Present address; 4) Degrees and other qualifications obtained, with name of University in each case; 5) Present professional activity; 6) List of publications; 7) Standard of knowledge of English and French — written and spoken; 8) Whether intending to stay in Naples unaccompained or with members of family and, in the former case, whether willing to share a room with other students. The applicant should send, together with his application, a note of reference from a University professor testifying the applicant's interest and preparation for the activities of the School.

Each application will be considered by the Director of the School on the basis of the information submitted, with regard also to a fair distribution of the places available among students of various nations. The decisions on the admittance to the School will be made known to the applicants by June 10th, 1961; in a few exceptional cases this date may be anticipated.

Students are requested to register at the Istituto di Fisica Teorica, Mostra d'Oltremare, Padiglione 19, in the afternoon of July 9th.

The total fee to be paid by students to the management of the School for attendance, hotel accommodation and full board is L. 40.000 if the student has single room accommodation and L. 35.000 if the student shares room accommodation with other students or with members of his family.

Fees should be paid not later than 5 days after the beginning of the course to the management of the School in Italian currency. A limited number of scholarships may be granted to students whose economic conditions might otherwise prevent them from attending the School.

The School will do everything possible to find suitable accomodation in local hotels for members of families accompanying the students. Members of student's families may avail themselves of the catering arrangements being organized for the School. All expenditures involved in the hotel accommodation, board etc. for students relatives will be payable separately. It is estimated that L. 3.500 per person per day may suffice.

For further information apply to the Director of the School, Professor Eduardo R. Caianiello, Istituto di Fisica Teorica, Mostra d'Oltremare, Pad. 19, Napoli.

The Director of the School
E. R. CAIANIELLO

IL NUOVO CIMENTO Vol. XX, N. 3 1° Maggio 1961

Effect of Inelastic Scattering on Polarization Asymmetry.

G. Felsner (*) and M. E. Rose

Oak Ridge National Laboratory - Oak Ridge, Tenn.

(ricevuto il 6 Febbraio 1961)

Summary. — The inelastic scattering of polarized relativistic electrons by atoms has been calculated in a modified first Born-approximation. In order to evaluate the matrix elements the Thomas-Fermi model of the atom has been used. The result shows that in the case of gold the asymmetry coming from the inelastic scattering makes a negligible change in the asymmetry to be attributed to the total, inelastic plus Mott, scattering.

1. – Introduction.

The measurement of the polarization of the electrons in nuclear β-decay has proved to be a useful tool for verification of the two component theory of the neutrino (1). The most accurate method for detection of the electron polarization is a Mott-scattering experiment where the asymmetry of the scattered electrons has to be measured. Sherman (2) has calculated the Mott-scattering asymmetry with an accuracy of about 1% and various experimental groups (3) have measured it with $(1 \div 5)\%$ accuracy. Such an accuracy, of course, requires a very careful discussion of possible errors and one of these might

(*) On leave from «Physikalisches Institut der Universität Erlangen».

(1) L. Landau: *Nucl. Phys.*, **3**, 127 (1957); A. Salam: *Nuovo Cimento*, **5**, 299 (1957); T. D. Lee and C. N. Yang: *Phys. Rev.*, **105**, 1671 (1957).

(2) N. Sherman: *Phys. Rev.*, **103**, 1601 (1956); N. Sherman and D. F. Nelson: *Phys. Rev.*, **114**, 1541 (1959).

(3) H. B. Willard, A. Galonsky, A. R. Brosi and B. H. Ketelle: to be published; H. Bienlein, G. Felsner, R. Fleischmann, K. Güthner, H. von Issendorff and H. Wegener: *Zeits. Phys.*, **154**, 376 (1959); **155** 101 (1959): for further references see: H. Schopper: *Fortschr. d. Phys.*, **8**, 327 (1960).

be the inelastic scattering of the electrons by the atoms which necessarily accompanies the nuclear scattering [1]. This effect has been neglected so far, presumably because it is difficult to make an accurate calculation of it. To be sure, the total inelastic scattering is small compared to the elastic contribution. But in the absence of evidence to the contrary one cannot conclude that the modification of the expected asymmetry is correspondingly small. In any event it is necessary to estimate the change in the asymmetry and to compare it with the quoted experimental accuracy.

The purpose of the present paper is to calculate the correction to the electron asymmetry from the inelastic scattering. Since the effect to be calculated is only a small correction, it is sufficient to look at it in an approximate way. Our numerical results show that the change of the asymmetry resulting from the inelastic scattering on gold atoms is only about 0.5 % of the Mott-scattering asymmetry and, therefore, it can be neglected easily.

2. – Calculations.

Inelastic scattering of electrons by atoms without taking into account polarization effects has been calculated thirty years ago by BETHE [5] and MOLLER [6]. Recently BATYGIN and TOPTYGIN [7] have calculated the inelastic scattering for polarized electrons, but since they use plane Dirac-waves and Born approximation the polarization of the scattered particle has to be measured also in order to see polarization effects.

In our problem we have to deal with the asymmetry in the scattering intensity resulting from the polarization of the incoming particles when only the direction of the scattered particle is measured. We start with the differential cross-section that Bethe has given for collisions of relativistic electrons with atoms. The electron excites the n-th energy level of the atom. Exchange effects are not taken into account.

$$(1) \qquad \mathrm{d}\sigma_n = 4\alpha^2 W W' \frac{p'}{p} \frac{|\boldsymbol{M}_0|^2}{[(\boldsymbol{p}-\boldsymbol{p}')^2 - (W-W')^2]^2} \mathrm{d}\Omega .$$

We use units in which $\hbar = m = c = 1$. α is Sommerfeld's fine structure constant. W, W' and p, p' are total energy and momentum before and after

[1] Effects of screening on elastic scattering are now being considered by N. SHERMAN (private communication to M. E. ROSE).

[5] H. BETHE: *Ann. d. Phys.*, **5**, 325 (1930) and *Zeits. Phys.*, **76**, 293 (1932).

[6] CH. MØLLER: *Zeits. Phys.*, **70**, 786 (1931) and *Ann. d. Phys.*, **14**, 531 (1932).

[7] V. V. BATYGIN and I. N. TOPTYGIN: *Soviet Physics JETP*, **10** (37), 975 (1960).

scattering, respectively. M_{n0} is the transition matrix element

$$M_{n0} = \int \Psi_n^* \sum_{j=1}^{z} \exp\left[i\boldsymbol{q}\cdot\boldsymbol{r}_j\right][a_0 - (\boldsymbol{a}\cdot\boldsymbol{\alpha}_j)]\Psi_0 \,\mathrm{d}\tau\,, \tag{2}$$

where Ψ_0 and Ψ_n are wave functions of the ground state and of the excited state of the atom. $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$ is the momentum transfer. The index j on the Dirac operator $\boldsymbol{\alpha}_j$ means that it acts on the j-th electron. The integral has to be taken over the space co-ordinates of all atomic electrons. The incident and the outgoing particle are given as solutions of the Dirac equation

$$\psi_{el} = u(\boldsymbol{p}) \exp\left[i(\boldsymbol{p}\cdot\boldsymbol{r} - Wt)\right]$$

and from these we get

$$a_0 = u_0^*(\boldsymbol{p}')\, u(\boldsymbol{p})\,, \qquad \boldsymbol{a} = u_0^*(\boldsymbol{p}')\boldsymbol{\alpha}\, u(\boldsymbol{p})\,. \tag{3}$$

In calculating the matrix elements we treat the atomic electrons in a non-relativistic approximation. We replace the relativistic current

$$\Psi_n^* \boldsymbol{\alpha}_j \Psi_0$$

by the Schrödinger current

$$\frac{1}{2i}\left(\Psi_n^* \nabla_j \Psi_0 - \Psi_0 \nabla_j \Psi_n^*\right).$$

For the incident (scattered) particle we take Dirac plane waves and modulate the amplitude in order to get the influence of the Coulomb field. Otherwise, of course, the asymmetry vanishes. It would be too complicated to take the complete solution of the Dirac equation in the Coulomb field and instead we take the isotropic part of the exact solution [8] at $r = 0$ as an amplitude of the plane waves. For the electron spinors $u(\boldsymbol{p})$ we take

$$u(\boldsymbol{p}) = \sum_m b_m u_m(\boldsymbol{p})\,,$$

where the $u_m(\boldsymbol{p})$ are spin eigenfunctions

$$u_m(\boldsymbol{p}) = \sqrt{\frac{W+\gamma}{2W}} \begin{bmatrix} \chi_m \\ M\,\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{W+1}\,\chi_m \end{bmatrix},$$

[8] J. D. Jackson, S. B. Treiman and H. W. Wyld: *Zeits. Phys.*, **150**, 640 (1958).

with

$$M = \frac{W+1}{W+\gamma}\left(1 + i\,\frac{(\alpha Z)}{p}\right), \qquad \gamma = \sqrt{1-(\alpha Z)^2}\,.$$

The b_m are determined by both, the direction of the polarization of the electrons

$$\boldsymbol{P} = \sum_{mm'} b_m^* b_{m'} (\chi_m, \boldsymbol{\sigma}\chi_{m'})$$

and the normalization condition

$$\sum_m |b_m|^2 = 1\,.$$

With that wave function we construct the density matrix

$$\{D(\boldsymbol{p})\}_{\varrho\lambda} = \{u(\boldsymbol{p})\}_\varrho \{u^*(\boldsymbol{p})\}_\lambda$$

for the incoming polarized electron. We get explicitly

$$D(\boldsymbol{p}) = \frac{1}{4}\Big\{1 + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \gamma\beta}{W} - i\,\frac{(\alpha Z)}{W}(\beta\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}) + \boldsymbol{P}\cdot\Big[\frac{\gamma}{W}\boldsymbol{\sigma} + \frac{W-\gamma}{W}(\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma})\hat{\boldsymbol{p}} - \tag{4}$$

$$+ \beta\boldsymbol{\sigma} - \frac{W-\gamma}{W}(\hat{\boldsymbol{p}}\cdot\beta\boldsymbol{\sigma})\hat{\boldsymbol{p}} - \frac{\gamma_5}{W}\boldsymbol{p} + \frac{i(\alpha Z)}{W}\beta\gamma_5\hat{\boldsymbol{p}}\Big] - \Big(\frac{i}{W}\beta\boldsymbol{\alpha} + \frac{(\alpha Z)}{pW}\boldsymbol{\alpha}\Big)\cdot[\boldsymbol{p}\times\boldsymbol{P}]\Big\}\,.$$

The scattered electrons have a density matrix

$$\{D_0(\boldsymbol{p})'\}_{\lambda\varrho} = \{u_0(\boldsymbol{p}')\}_\lambda \{u_0^*(\boldsymbol{p}')\}_\varrho\,,$$

where the index 0 indicates, that we have made an average over all polarization directions,

$$D_0(\boldsymbol{p}') = \frac{1}{2}\Big\{1 + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}' + \gamma\beta}{W'} - \frac{i(\alpha Z)}{W'}(\beta\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}')\Big\}\,. \tag{5}$$

We now write the matrix element M_{n0} as follows

$$M_{n0} = a_0 F_{n0} - (\boldsymbol{a}\cdot\boldsymbol{G}_{n0})\,, \tag{6}$$

with

$$F_{n0} = \sum_j \int \Psi_n^* \exp[i\boldsymbol{q}\cdot\boldsymbol{r}_j]\Psi_0\,\mathrm{d}\tau \equiv I_0\,, \tag{7}$$

and

$$\boldsymbol{G}_{n0} = \frac{1}{2i}\sum_j \int \exp[i\boldsymbol{q}\cdot\boldsymbol{r}_j](\Psi_n^*\boldsymbol{\nabla}_j\Psi_0 - \Psi_0\boldsymbol{\nabla}_j\Psi_n^*)\,\mathrm{d}\tau\,. \tag{8}$$

This can as well be written as

$$\boldsymbol{G}_{n0} = \frac{1}{2i} \sum_j \left\{ 2 \int \exp\left[i\boldsymbol{q}\cdot\boldsymbol{r}_j \right] \Psi_n^* \nabla_j \Psi_0 \, \mathrm{d}\tau + i\boldsymbol{q} \int \exp\left[i\boldsymbol{q}\cdot\boldsymbol{r}_j \right] \Psi_n^* \Psi_0 \, \mathrm{d}\tau \right\},$$

or

$$\boldsymbol{G}_{n0} = \frac{1}{2i} \left\{ 2\boldsymbol{I}_1 + i\boldsymbol{q} I_0 \right\}.$$

Now we use the projection operators $D(\boldsymbol{p})$ and $D_0(\boldsymbol{p}')$ and get for the absolute value square of the matrix element

$$(9) \qquad |M|^2 = |F|^2 \operatorname{Tr}\{D_0 D\} - 2 \operatorname{Re} F^* \operatorname{Tr}\{D_0(\boldsymbol{\alpha}\cdot\boldsymbol{G}) D\} + \operatorname{Tr}\{D_0(\boldsymbol{\alpha}\cdot\boldsymbol{G}) D(\boldsymbol{\alpha}\cdot\boldsymbol{G}^*)\}.$$

For simplicity we have dropped the indices.

If one works out the traces one finds that $|M|^2$ depends on the following expressions

$$(a) \quad I_0 I_0^*, \qquad (b) \quad \boldsymbol{I}_1 I_0^*, \qquad (c) \quad \boldsymbol{I}_1 \bigcirc \boldsymbol{I}_1^*,$$

where the $\bigcirc$ in (c) indicates either one of the four products

$$(\boldsymbol{I}_1\cdot\boldsymbol{I}_1^*); \quad [\boldsymbol{I}_1\times\boldsymbol{I}_1^*]; \quad (\boldsymbol{I}_1\cdot\boldsymbol{A})(\boldsymbol{I}_1^*\cdot\boldsymbol{B}); \quad [\boldsymbol{I}_1\times\boldsymbol{A}](\boldsymbol{I}_1^*\cdot\boldsymbol{B}).$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are three-dimensional vectors like $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{q}$.

In order to evaluate this expression we follow a method of Heisenberg [9]. First, we make the assumption that the wave function of the atom can be written as an antisymmetric product of one-electron eigenfunctions. Secondly, we sum up all possible contributions from the excitations of the atom. We use closure to do this summation. When we now take only the inelastic contributions we get

$$(10) \quad \begin{cases} I_0 I_0^* = Z - 2 \iint |\sum_s \psi_s^*(r)\psi_s(r')|^2 \exp\left[i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}') \right] \mathrm{d}\tau \, \mathrm{d}\tau', \\ \boldsymbol{I}_1 I_0^* = -2 \iint \left(\sum_t \psi_t^*(r') \nabla\psi_t(r) \right) \left(\sum_s \psi_s^*(r)\psi_s(r') \right) \exp\left[i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}') \right] \mathrm{d}\tau \, \mathrm{d}\tau', \\ I_1 \bigcirc I_1^* = \int \sum_s \left(\nabla\psi_s^*(r) \right) \bigcirc \left(\nabla\psi_s(r) \right) \mathrm{d}\tau - \\ - 2 \iint \left[\sum_t \left(\nabla\psi_t^*(r') \right) \bigcirc \left(\nabla\psi_t(r) \right) \right] \left(\sum_s \psi_s^*(r)\psi_s(r') \right) \exp\left[i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}') \right] \mathrm{d}\tau \, \mathrm{d}\tau', \end{cases}$$

[9] W. Heisenberg: *Phys. Zeitschr.*, **32**, 737 (1931).

where the ψ now are single-electron eigenfunctions and the summations are to be taken over all occupied one-electron states in the atom (ground state).

Using the Thomas-Fermi model we write

$$\sum_s \psi_s^*(r)\psi_s(r') = \frac{1}{(2\pi)^3}\int_0^{p_0} \exp\left[-i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')\right]\mathrm{d}^3\boldsymbol{p}\,, \tag{11}$$

where the maximum momentum of the electrons in the atom, p_0, is connected with the potential V through

$$p_0^2 = 2V\left(\frac{r+r'}{2}\right). \tag{12}$$

For convenience we use here the atomic units $\hbar = m = e = 1$. Instead of $\boldsymbol{r}$ and $\boldsymbol{r}'$ we employ a new pair of variables

$$\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{r}', \qquad \boldsymbol{r}_2 = \frac{\boldsymbol{r}-\boldsymbol{r}'}{2}.$$

We insert (11) and similar expressions into (10) and perform the integration over $\mathrm{d}\tau_1$ as well as over the momentum space. The results are

$$\left\{\begin{aligned}
&I_0 I_0^* = Z - 2J_0\,,\\
&\boldsymbol{I}_1 I_0^* = -2i\hat{\boldsymbol{q}}J_0\alpha\,,\\
&(\boldsymbol{I}_1\cdot\boldsymbol{I}_1^*) = 3J_0^0 - 2J_2^0)\alpha^2\,,\\
&[\boldsymbol{I}_1\times\boldsymbol{I}_1^*] = 0\,,\\
&(\boldsymbol{I}_1\cdot\boldsymbol{A})(\boldsymbol{I}_1^*\cdot\boldsymbol{B}) = \left\{(\boldsymbol{A}\cdot\boldsymbol{B})J_0^0 - 2\left[(\boldsymbol{A}\cdot\boldsymbol{B})J_2^1 - (\hat{\boldsymbol{q}}\cdot\boldsymbol{A})(\hat{\boldsymbol{q}}\cdot\boldsymbol{B})J_2^2\right]\right\}\alpha^2\,,\\
&[\boldsymbol{I}_1\times\boldsymbol{A}](\boldsymbol{I}_1^*\cdot\boldsymbol{B}) \quad \{[\boldsymbol{B}\times\boldsymbol{A}]J_0^0 - 2([\boldsymbol{B}\times\boldsymbol{A}]J_2^1 - [\hat{\boldsymbol{q}}\times\boldsymbol{A}](\hat{\boldsymbol{q}}\cdot\boldsymbol{B})J_2^2)\}\alpha^2\,.
\end{aligned}\right. \tag{13}$$

The α appears because of the use of atomic units in terms of which the J are calculated. The J_n and J_n^m stand for the following integrals over $\mathrm{d}\tau_2$ which is the only integration that has to be done numerically.

$$\left\{\begin{aligned}
&J_0 = \frac{1}{6\pi^2}\int\left(p_0 - \frac{q}{2}\right)^2\left(p_0 + \frac{q}{4}\right)\mathrm{d}\tau_2\,,\\
&J_1 = \frac{1}{8\pi^2}\int q^2\left(p_0 - \frac{q}{2}\right)^2\mathrm{d}\tau_2\,,\\
&J_0^0 = \frac{1}{30\pi^2}\int p_0^5\,\mathrm{d}\tau_2\,,
\end{aligned}\right. \tag{14}$$

$$
(14)\quad \left\{
\begin{aligned}
J_2 &= \frac{1}{16\pi^2}\int\left\{\frac{9}{5}\,p_0^5 - p_0^4 q + \frac{1}{5}\,(q-p_0)^5\right\}\mathrm{d}\tau_2\,,\\
J_2^1 &= \frac{1}{8\pi^2}\int\left\{\frac{1}{4}\,p_0^4(2p_0-q) - \frac{1}{6}\,p_0^2\left[p_0^3-(q-p_0)^3\right] + \right.\\
&\qquad\qquad \left. + \frac{1}{20}\left[p_0^5-(q-p_0)^5\right]\right\}\mathrm{d}\tau_2\,,\\
J_2^2 &= \frac{1}{8\pi^2}\int\left\{\frac{1}{4}\,p_0^4(2p_0-q) - \frac{1}{2}\,p_0^2\left[p_0^3-(q-p_0)^3\right] + \right.\\
&\qquad\qquad \left. + \frac{1}{4}\left[p_0^5-(q-p_0)^5\right]\right\}\mathrm{d}\tau_2\,.
\end{aligned}
\right.
$$

In these integrals p_0 is given by the potential $V(r_2)$ and this is connected with the Thomas-Fermi potential $\varphi(x)$ as follows:

$$
(15)\qquad r_2 V(r_2) = Z\,\varphi(x)\,,
$$

with

$$
(16)\qquad r_2 = \frac{1}{2}\left(\frac{3\pi}{4}\right)^{\frac{2}{3}}\frac{x}{Z^{\frac{1}{3}}}\,,
$$

and this gives us

$$
(17)\qquad p_0 = 2\left(\frac{4}{3\pi}\right)^{\frac{1}{3}} Z^{\frac{2}{3}}\sqrt{\frac{\varphi(x)}{x}}\,.
$$

With these integrals we can now write down an explicit expression for the matrix element summed over all possible transitions

$$
\begin{aligned}
(18)\qquad \sum_n |M_{n0}|^2 &= A_1\left\{WW' + \gamma^2 + \left(1-\frac{(\alpha Z)^2}{pp'}\right)(\boldsymbol{p}\cdot\boldsymbol{p}')\right\} + \\
&+ \alpha A_2\left\{W'(\boldsymbol{p}\cdot\hat{\boldsymbol{q}}) + W(\boldsymbol{p}'\cdot\hat{\boldsymbol{q}})\right\} + \alpha^2 A_3\left\{WW' + \gamma^2 - \left(1-\frac{(\alpha Z)^2}{pp'}\right)(\boldsymbol{p}\cdot\boldsymbol{p}')\right\} + \\
&+ 2\alpha^2 A_4\left(1-\frac{(\alpha Z)^2}{pp'}\right)(\boldsymbol{p}\cdot\boldsymbol{p}') + 2\alpha^2 A_5\left(1-\frac{(\alpha Z)^2}{pp'}\right)(\boldsymbol{p}\cdot\hat{\boldsymbol{q}})(\boldsymbol{p}'\cdot\hat{\boldsymbol{q}}) + \\
&+ \boldsymbol{P}\cdot\left\{\left[A_1 + \alpha^2(A_4 - 2A_5)\right](p+p') + \alpha A_2(W-W')\right\}(\alpha Z)[\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}']\,,
\end{aligned}
$$

with

$$
\left\{
\begin{aligned}
A_1 &= \frac{1}{2WW'}\,(Z-2J_0)\,,\\
A_2 &= \frac{1}{2WW'}\,[4J_1 - q(Z-2J_0)],\\
A_3 &= \frac{1}{2WW'}\left[3J_0^0 - 2J_2^0 - 2qJ_1 + \frac{1}{4}\,q^2(Z-2J_0)\right],\\
A_4 &= \frac{1}{2WW'}\,(J_0^0 - 2J_2^1)\,,\\
A_5 &= \frac{1}{2WW'}\left[2J_2^2 - 2qJ_1 + \frac{1}{4}\,q^2(Z-2J_0)\right].
\end{aligned}
\right. \tag{19}
$$

In (18) the term $\boldsymbol{P}\cdot[\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}']$ corresponds to the scattering asymmetry arising from the polarization of the electrons. Numerical values for gold show that the terms with A_3, A_4, A_5 contribute less than one percent to the whole sum (18). We leave these small terms out when we write the differential cross-section. This is in units of $(\hbar/mc)^2$

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 4\alpha^2 WW'\,\frac{p'}{p}\,\frac{\varkappa_0 + \varkappa_1(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') + \varkappa_2\boldsymbol{P}\cdot[\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}']}{[(\boldsymbol{p}-\boldsymbol{p}')^2 - (W-W')^2]^2}\,, \tag{20}
$$

where

$$
\left\{
\begin{aligned}
\varkappa_0 &= A_1(WW' + \gamma^2) + \alpha A_2(W'p - Wp')\,,\\
\varkappa_1 &= A_1(pp' - (\alpha Z)^2) - \alpha A_2(W'p - Wp')\,,\\
\varkappa_2 &= (\alpha Z)[A_1(p+p') + \alpha A_2(W - W')]\,.
\end{aligned}
\right. \tag{21}
$$

For transversely polarized electrons which are scattered through an angle ϑ, $((\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = \cos\vartheta)$, the cross-section (20) gives the following asymmetry:

$$
S_{\mathrm{incl}}(\vartheta) = \frac{\varkappa_2 \sin\vartheta}{\varkappa_0 + \varkappa_1\cos\vartheta}\,. \tag{22}
$$

This asymmetry is, for gold, opposite in sign and about half as big as the Mott-scattering asymmetry. However, because the inelastic cross-section is less than 0.01 times the Mott cross-section, the net effect of the inelastic scattering on the asymmetry is about one-half percent.

From Fig. 1 and 2 one can see that the energy dependence does not change this conclusion: that the asymmetry which comes from inelastic scattering

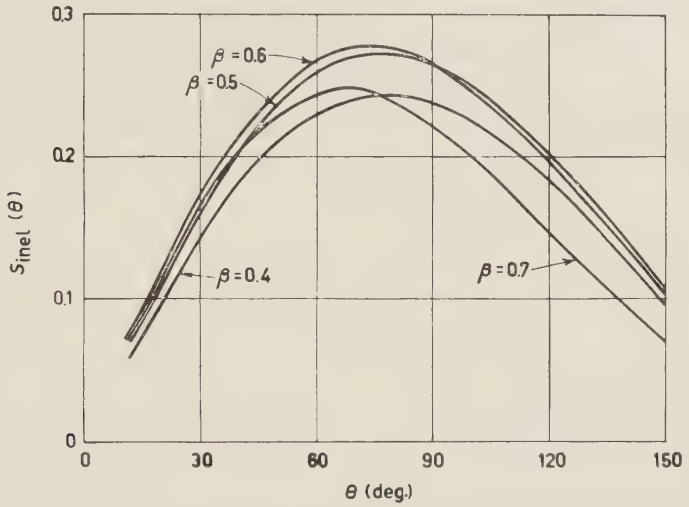


Fig. 1. – The asymmetry associated with the inelastic scattering of transversely polarized electrons on gold atoms *vs.* the scattering angle for different energies. We have made the approximation $W' \approx W$, that is, energy loss small compared to the incident energy.

makes a correction of only 0.5% on the Mott scattering asymmetry. Since neither theory nor experiment are at present more accurate than one percent,

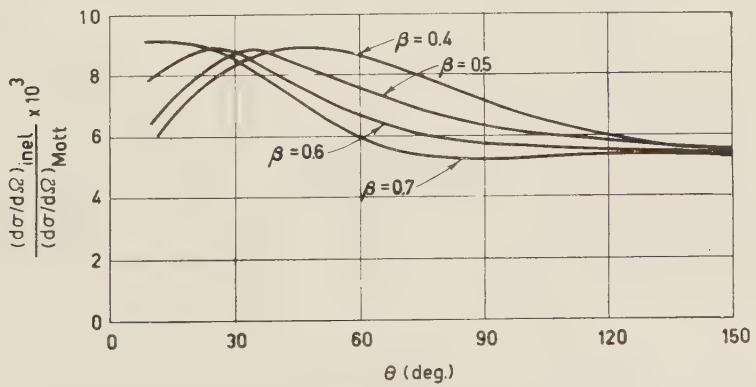


Fig. 2. – The ratio inelastic differential cross section to Mott differential cross-section for gold and different energies *vs.* scattering angle under the same assumption as in Fig. 1.

no correction at all need be made for inelastic scattering in the polarization analysis experiments.

* * *

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RIASSUNTO (*)

Abbiamo calcolato, nella prima approssimazione di Born modificata, lo scattering anelastico su atomi degli elettroni relativistici polarizzati. Allo scopo di valutare gli elementi di matrice abbiamo usato il modello atomico di Thomas-Fermi. I risultati mostrano che, nel caso dell'oro, l'asimmetria derivante dallo scattering anelastico produce un cambiamento trascurabile all'asimmetria attribuibile alla somma degli scattering anelastico e di Mott.

(*) *Traduzione a cura della Redazione.*

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Evidence for Pion-Pion Interactions from s-Wave Pion-Nucleon Scattering (*).

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(ricevuto il 6 Febbraio 1961)

Summary. — By fitting the s-wave partial amplitudes for π-$\mathcal{N}$-scattering, in an unphysical region, we have greatly extended the energy range over which we can examine the contribution from the process $\pi+\pi\to\mathcal{N}+\overline{\mathcal{N}}$. Our method shows up contributions from this process in states of isotopic spin $T=0$ and $T=1$. We can estimate the energies $t^{\frac{1}{2}}$ for which this process is important. The form of these contributions is just what would be expected from considerations of angular momenta. Estimates of the amplitudes for $\pi+\pi\to\mathcal{N}+\overline{\mathcal{N}}$ are deduced. The $T=0$ amplitude is large, but in the $T=1$ case the amplitude is much smaller than the values which have been predicted from the nucleon isovector form factors.

1. – Introduction.

In another paper hereafter referred to as I, two of us (J.H. and T.D.S.) [1] gave an account of the evaluation of the dispersion relations for the s-wave partial amplitudes $f_0^{(T)}(s)$ in pion-nucleon scattering. ($T=\frac{1}{2}, \frac{3}{2}$ is the isospin and $s=[(M^2+q^2)^{\frac{1}{2}}+(\mu^2+q^2)^{\frac{1}{2}}]^2$ is the energy variable. M, μ, are the nucleon and pion masses and q is the c.m. momentum). The relations are

$$\text{(1)}\qquad \operatorname{Re} f_0^{(T)}(s) = (\text{Born terms})^{(T)} + \frac{1}{\pi} P\int_{(M+\mu)^2}^{\infty} \frac{\operatorname{Im} f_0^{(T)}(s')}{s'-s}\,\mathrm{d}s' + \frac{1}{\pi}\int_0^{(M-\mu)^2} \frac{\operatorname{Im} f_0^{(T)}(s')}{s'-s}\,\mathrm{d}s' + \Delta^{(T)}(s)\,.$$

(*) This work was supported in part by a grant from the U. S. Air Force, European Office, Air Research and Development Command.

(1) J. Hamilton and T. D. Spearman: *Ann. of Phys.* **12**, 172 (1961).

The second and third terms on the right are the *physical integral* (re-scattering) and the *crossed integral*; they come from the cuts $(M+\mu)^2 \leqslant s < \infty$, $0 \leqslant s \leqslant (M-\mu)^2$ respectively (Fig. 1). From the known values of the phase shifts and the coupling constant f^2, all terms in (1) can be evaluated except $\Delta^{(T)}(s)$. This is the *discrepancy*, and gives the contribution to $\mathrm{Re}\, f_0^{(T)}(s)$ from the cut $-\infty < s \leqslant 0$ and the circle $|s| = M^2 - \mu^2$ (Fig. 1); the cut $-\infty < s \leqslant 0$ corresponds to effects whose range is $\hbar/\mu c$ or less, and the circle arises from the process $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$. The right-hand portion of the circle can give comparatively long range effects in π-$\mathcal{N}$ s-wave scattering. These should show up in an appreciable variation of $\Delta^{(T)}(s)$ with s at low energies.

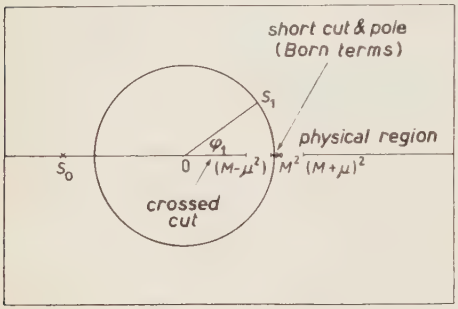

Fig. 1. Poles and branch cuts of the π-$\mathcal{N}$, s-wave scattering amplitude $f_0^{(T)}(s)$, in the complex s-plane.

A sizable contribution from the right-hand portion of the circle implies an appreciable imaginary part of the amplitude for $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$, which in turn implies appreciable pion-pion scattering ($\pi+\pi \to \pi+\pi$) at fairly low energies ([2]). It is convenient to use

$$\Delta^{(+)} = \tfrac{1}{3}(\Delta^{(\frac{1}{2})} + 2\Delta^{(\frac{3}{2})}), \qquad \Delta^{(-)} = \tfrac{1}{3}(\Delta^{(\frac{1}{2})} - \Delta^{(\frac{3}{2})}),$$

$\Delta^{(+)}$, $\Delta^{(-)}$ correspond to the isospin values $T=0$, 1, respectively, for the $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$ system. FRAZER and FULCO ([3]) have suggested that a resonance in the $T=1$ state of $\pi+\pi \to \pi+\pi$ at $t_R = 4(\mu^2+\bar{q}^2) \simeq \simeq 11.5\mu^2$ will explain the iso-

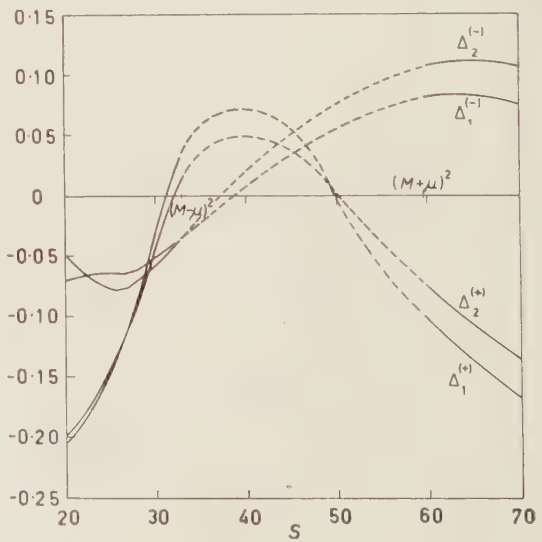

Fig. 2. – The discrepancies $\Delta_{1,2}^{(\pm)}(s)$, in units $\hbar=\mu=c=1$. The subscripts 1, 2 refer to the values of $\Delta^{(\pm)}(s)$ derived by using the sets of phase shifts (a_1) and (a_2) described in Section 2.

([2]) W. R. FRAZER and J. R. FULCO: *Phys. Rev.*, **117**, 1603 (1960).
([3]) W. R. FRAZER and J. R. FULCO: *Phys. Rev.*, **117**, 1609 (1960).

vector form factors of the nucleon ($\bar{q}$ is the c.m. momentum for $\pi+\pi \to \to \pi+\pi$).

In I the values of $\Delta^{(\pm)}(s)$ were determined from the threshold, $s=(M+\mu)^2= = 59.6$, up to $s=70$ (*i.e.*, 110 MeV lab. energy). We use the units $\hbar=c= =\mu=1$ throughout. The results are shown on the right of Fig. 2; by themselves these give no evidence for a $T=1$ π-π interaction, but the strong energy dependence of $\Delta^{(+)}$ suggests that there is a long range π-π interaction in the $T=0$ state.

2. – The crossing method.

We have evaluated $\Delta^{(\pm)}(s)$ in the unphysical region $s\leqslant(M-\mu)^2=32.7$ to get much better information about the circle contribution. This enables us to determine the discrepancy over a much greater energy range. Our knowledge of the π-$\mathcal{N}$ phase shifts is sufficiently good to do this for $21<s<32.7$. The values of $\mathrm{Re}\, f_0^{(T)}(s)$ in this region are found in terms of the known π-$\mathcal{N}$ scattering amplitudes in the physical region $s\geqslant(M+\mu)^2$, by using the crossing relations:

$$A^{(\pm)}(s,t,\bar{s})=\pm A^{(\pm)}(\bar{s},t,s)\;;\qquad B^{(\pm)}(s,t,\bar{s})=\mp B^{(\pm)}(\bar{s},t,s)\,.$$

The analogous determination of $\mathrm{Im}\, f_0^{(T)}(s)$ for $0\leqslant s\leqslant(M-\mu)^2$ is discussed in I. In the present calculation however we have taken account of s-, p-, and d-waves. The third term on the right of (1) now becomes a Cauchy principal value integral, and $\mathrm{Im}\, f_0^{(T)}(s)$ has to be determined accurately in $21\leqslant s\leqslant 32.7$. The results of using eq. (1) in this way are plotted in Fig. 2.

We comment briefly on some points which arise in this calculation.

a) There is some doubt about the value of the $T=\frac{1}{2}$ s-wave phase shift α_1 above 120 MeV. We have therefore taken two extreme cases. In set (*a*1) α_1 reaches a fairly large positive maximum about 600 MeV [4]. In set (*a*2) α_1 becomes negative about 350 MeV and stays negative at higher energies [5]. The values of $\mathrm{Re}\, f_0^{(T)}(s)$ and the crossed integral as deduced from these two sets differ little in the range $21\leqslant s\leqslant 32.7$. Hence this ambiguity in the value of α_1 causes only a small error in $\Delta^{(\pm)}(s)$ cf. (Fig. 2).

b) For high energies all partial waves which we require are assumed to be highly inelastic. The partial wave amplitudes are continued smoothly from

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([4]) W. D. Walker, J. Davis and W. D. Shephard: *Phys. Rev.*, **118**, 1612 (1960).

([5]) This is similar to the behaviour of α_1 in solution «*b*/*spd*» of S. M. Korenchenko, N. I. Polumordvinova, G. T. Tentiukova and G. V. Zinov: *Soviet Physics JETP*, **11**, 1016 (1960). It agrees with the curve for $(\alpha_1-\alpha_3)$ obtained using dispersion theory by A. C. Finn: *Phys. Rev.*, **119**, 1786 (1960).

the known values to behave as $1/2q$ at high energies. Any error in this procedure will give rise to an additive *constant* in our values of $A^{(+)}(s)$ or $A^{(-)}(s)$.

c) The value of $\mathrm{Im}\, f_0^{(T)}(s)$ as $s \to 0$ is related to the backward π-$\mathcal{N}$ scattering at very high (physical) energies, as was shown in I. We make two alternative assumptions here: (*c*1) The total π-$\mathcal{N}$ amplitude $f^{(-)}(\theta = \pi, \omega)$ behaves as $1/\omega$ when the pion energy $\omega \to \infty$, giving $\mathrm{Im}\, f_0^{(-)}(s) \sim 1/\sqrt{s}$ for small s; (*c*2) $\mathrm{Im}\, f_0^{(+)}(s_3) \to 0$ as $s_3 \to 0$. Arguments may be advanced to show that $f^{(+)}(\theta = \pi, \omega)$ is not of order greater than $1/\omega$ as $\omega \to \infty$ and so (*c*1) and (*c*2) represent the possible extremes in behaviour. We have assumed that the total π-$\mathcal{N}$ amplitude $f^{(-)}(\theta = \pi, \omega) \to 0$ as $\omega \to 0$. This follows from the assumption that scattering is independent of isotopic spin at high energies [6]. In both cases, (*c*1) and (*c*2), we fit the corresponding asymptotic behaviour of $\mathrm{Im}\, f_0^{(T)}(s)$ (as $s \to 0$) on to the values of $\mathrm{Im}\, f_0^{(T)}(s)$ which are determined down to fairly small values of s by the known phase shifts. The curves for $A^{(-)}(s)$ for cases (*c*1) and (*c*2) lie close together and have almost the same energy dependence. In Fig. 2 we show the mean of these two curves.

3. – Discussion of the results.

It is reasonable to assume that $A^{(+)}(s)$ and $A^{(-)}(s)$ have a smooth behaviour [7] in the region $32 \leqslant s \leqslant 60$; this is shown in Fig. 2. Because $A^{(-)}(s)$ arise from the circle and the cut $-\infty < s \leqslant 0$ we attempt to represent these functions by the sum of three pole terms:

$$A^{(\pm)}(s) \quad \frac{b_\pm}{s_0 - s} + \frac{c_\pm}{s_1 - s} + \frac{c_\pm^*}{s_1^* - s}, \tag{2}$$

where s_0 is on the negative real axis; s_1, s_1^* lie on or close to the circle.

The constants $b_\pm$ are real and the constants $c_\pm$ may be complex. We may write (2) in the form

$$A^{(\pm)}(s) = \frac{-2\,\mathrm{Re}\, c_\pm \cdot (s - \mathrm{Re}\, s_1)}{(s - \mathrm{Re}\, s_1)^2 + (\mathrm{Im}\, s_1)^2} + \frac{2\,\mathrm{Im}\, c_\pm \cdot \mathrm{Im}\, s_1}{(s - \mathrm{Re}\, s_1)^2 + (\mathrm{Im}\, s_1)^2} + \frac{b_\pm}{s_0 - s}. \tag{3}$$

The results obtained by fitting the values of $A^{(\pm)}(s)$ shown in Fig. 2 with eq. (3) are summarized in Table I.

(6) See, for example, I. IA. POMERANCHUK: *Žurn. Èksp. Teor. Fiz.*, **34**, 725 (1958).

(7) $A^{(+)}(s)$ is given by a dispersion integral over the discontinuities across the cut $-\infty < s \leqslant 0$ and across the circle. The discontinuity across the circle vanishes as $s \to M^2 - \mu^2$. Therefore $A^{(\pm)}(s)$ should behave smoothly on $(M - \mu)^2 \leqslant s \leqslant (M + \mu)^2$.

We see that c_- is almost real and c_+ is almost imaginary. In Section 5 we shall show that this is what we would expect from simple considerations of the angular momentum states involved.

TABLE I. - $\Delta_1^{(\pm)}(s)$ are the values of the discrepancy calculated with the phase shift set (a_1); $\Delta_2^{(\pm)}(s)$ are calculated with the set (a_2). $s_1 = |s_1| \exp[i\varphi_1]$ are the positions of the poles on the circle, and $c_\pm$ are the residues. s_0 is the position of the pole on the negative real axis and b is its residue.

	$\Delta_1^{(+)}$	$\Delta_2^{(+)}$	$\Delta_1^{(-)}$	$\Delta_2^{(-)}$
Re s_1	36	36	38	40
Im s_1	25.5	25.5	22.5	19
φ_1	33.5	33.5°	31°	26°
t_{max}	$18.7\mu^2$	$18.7\mu^2$	$16.6\mu^2$	$12.9\mu^2$
Re c	0	— 0.9	— 1.73	— 1.71
Im c	8.1	6.4	— 0.29	— 0.53
b	42	42	—	—
s_0	— 35	— 63	—	—
$\dfrac{b}{s_0 - s}\Big\vert_{s=(M+\mu)^2}$	— 0.44	— 0.38	0.02	0.05

The positions of the poles s_1, s_1^* enable us to make some deductions about the values of t for which the process $\pi + \pi \to \mathcal{N} + \overline{\mathcal{N}}$ contributes appreciably to π-$\mathcal{N}$ scattering. If θ is the π-$\mathcal{N}$ scattering angle, then for $s_1 = |s_1| \exp[i\varphi_1]$,

$$t = 4 \sin^2\left(\frac{\theta}{2}\right)\left\{\mu^2 \cos^2\frac{\varphi_1}{2} + M^2 \sin^2\frac{\varphi_1}{2}\right\}.$$

Hence the values of t contributing to $\Delta^{(\pm)}(s)$ are $t_{max} \sin^2\theta/2$ $(0 \leqslant \theta \leqslant \pi)$, where $t_{max} = 4(\mu^2 \cos^2(\varphi_1/2) + M^2 \sin^2(\varphi_1/2))$.

Therefore our results are consistent with the existence of maxima or minima in the imaginary parts of the amplitudes for $\pi + \pi \to \mathcal{N} + \overline{\mathcal{N}}$ in the $T = 0$, $T = 1$ states, for values of t not greater than $19\mu^2$ and $17\mu^2$ respectively. At the least, our results show that these amplitudes must be appreciable for $t \leqslant 19\mu^2$ and $t \leqslant 17\mu^2$ respectively [8].

The terms $b_\pm/(s_0 - s)$ represent a residual short range π-$\mathcal{N}$ interaction. We see immediately that $|b_-/(s_0 - s)| \ll |b_+/(s_0 - s)|$ near the low energy physical region. Hence this short range contribution is almost the same in the $T = \frac{1}{2}$

[8] Cf. J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Nuovo Cimento*, **16**, 918 (1960); *Phys. Rev. Lett.*, **5**, 386 (1960); *Nuovo Cimento*, **19**, 142 (1961).

and $T=\frac{3}{2}$ π-$\mathcal{N}$ states and is well approximated by the term $b_+(s_0-s)$ for scattering energies up to about 100 MeV; its value is approximately -0.4.

The only other appreciable short range contribution to $\mathrm{Re}\, f_0^{(T)}(s)$ comes from the high energy terms in the physical integral. From Tables I and II of I we see that this gives no more than 0.1 (again it is much the same in the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states)

We conclude that the short range forces in the π-$\mathcal{N}$ scattering, in both the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states, are equivalent to a repulsive «hard core» whose radius is about 0.3 $(\hbar/\mu c)$; this is because by themselves these contributions give $a_1 \simeq a_3 \simeq -0.3$. We note that the hard core radius 0.3 $(\hbar/\mu c)$ is surprisingly large [9].

The $T=0$ π-π interaction gives a positive contribution to $\mathrm{Re}\, f_0^{(T)}(s)$ for $s \geqslant (M+\mu)^2$; therefore it has an attractive effect on low energy π-$\mathcal{N}$ scattering. From the figures in the Table we see that this attraction is strong.

4. – The δ-function approximation and the nucleon form factors.

For a sharp $T=1$ state π-π resonance at $t=t_R$, we may write

$$\mathrm{Im}\, \Gamma_1(t) = \gamma_1\, \delta(t-t_R)\,, \qquad \mathrm{Im}\, \Gamma_2(t) = (\gamma_2/M)\cdot\delta(t-Y\, t_R)\,, \qquad (t>4\mu^2),$$

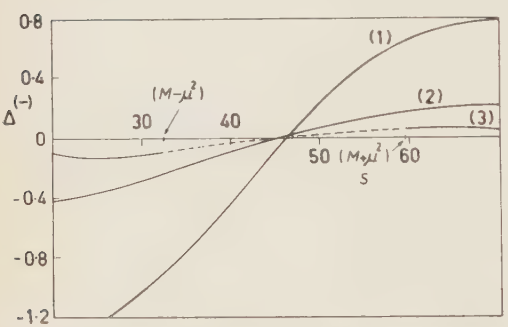

Fig. 3. Curve (1) shows the contribution to $\Delta^{(-)}$ corresponding to a resonance of the form suggested by FRAZER and FULCO, with $t_R = 11.5$, $\gamma_1 = -6.9$, $\gamma_2 = -15.6$; curve (2) corresponds to the resonance used by BOWCOCK, COTTINGHAM and LURIÉ, with $t_R = 22.4$, $\gamma_1 = -3.1$, $\gamma_2 = -5.6$; curve (3) shows for comparison the results obtained by our method.

where $\Gamma_1(t)$, $\Gamma_2(t)$ are related to the nucleon isovector form factors. FRAUTSCHI and WALECKA [10], by fitting the form factors, obtain the values

$$\gamma_1 = -6.9\,, \qquad \gamma_2 = -15.6\,.$$

They assume the resonance solution for the $T=1$ state $\pi+\pi\to\pi+\pi$ amplitude obtained by FRAZER and FULCO [3], with $t_R = 11.5$. The discontinuity across the circle may be calculated in terms of γ_1 and γ_2 and so the circle contribution due to Frazer and Fulco's resonance may be evaluated. The result

[9] The results of fitting the $A^{(+)}(s)$ curve, as quoted in the Table, depend somewhat noticeably on the form of $A^{(+)}(s)$ in the unphysical region $s<32.7$. Small errors in this region can give an appreciable change in the results. However, it seems unlikely that Im c_+ can be less than 4.0; the corresponding value of $(s_0-s)/b_+$ is -0.3 and the hard core radius becomes $0.2\hbar/\mu c$.

[10] S. C. FRAUTSCHI and J. D. WALECKA: *Phys. Rev.*, **120**, 1486 (1960).

is shown in Fig. 3, where it is compared with our values for the circle contribution. It is seen that the circle contribution calculated in this way has the correct sign but γ_1 turns out to be too large by a factor ~ 10. Fitting our results for $\Delta^{(-)}$ with a resonance at $t_R = 11.5\,\mu^2$ requires a value for $\gamma_1 \simeq -0.6$. This agrees with Frautschi's [11] investigation of the low energy behaviour of the amplitude for π-$\mathcal{N}$ scattering in the $p_{\frac{1}{2}}$, $T=\frac{1}{2}$ state; he gets results which are too large by a factor of 5 or more.

Bowcock *et al.* [8] obtain the result $\gamma_2/M\gamma_1 = 0.27/\mu$ from the nucleon form factors. Also they use $t_R = 22.4\,\mu^2$; this increase in t_R helps to reduce the effect of the π-π interaction on π-$\mathcal{N}$ scattering. Fitting the energy dependence of the s-wave π-$\mathcal{N}$ scattering in the physical range, these authors get $\gamma_1 \simeq -3.1$ (with $t_R = 22.4\,\mu^2$). The circle contribution corresponding to these values of γ_1, γ_2, t_R is also shown in Fig. 3. A resonance at $t_R = 22.4\,\mu^2$ would require a value of $\gamma_1 \simeq -1.0$ to give agreement with our results. (In this case γ_1 must be reduced by a factor of 3 from Bowcock *et al.*'s result.)

Now t_R should be less than our values for t_{max} for the $\Delta^{(-)}(s)$ case. It is therefore unlikely that t_R can be as large as $22.4\,\mu^2$. As we saw above, for $t_R = 11.5\,\mu^2$, the $\Delta^{(-)}(s)$ curve requires $\gamma_1 \simeq -0.6$. In I we commented [12] that this disagreement between the isovector nucleon form factors and the low energy pion scattering may partly arise from inaccuracies in the calculation [3] of $\Gamma_1(t)$. Recently an improved calculation by Ball and Wong [13] has produced no substantial change in $\Gamma_1(t)$.

The small amplitude we obtain for the ($T=1$ state) process $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$ for small values of $t > 4\,\mu^2$ suggests that the process $\pi+\pi \to \pi+\pi$ cannot have a sharp resonance at low energies in the $T=1$ state. However, our results do show that the phase shift δ_1 for the latter process must depart appreciably from zero for some values of t less than $15\,\mu^2$. δ_1 is positive, corresponding to an attractive π-π interaction in the $T=1$ state.

5. – The form of $c_\pm$ and the angular momentum of $\pi+\pi \to \mathcal{N}+\mathcal{N}$.

We now examine the reality conditions for $c_\pm$ which arise because the equivalent poles s_1, s_1^* in both cases have fairly small phase angles φ_1. The contribution of the circle to $\mathrm{Re}\, f_0^{(T)}(s)$ is

$$\frac{1}{2\pi i}\oint \frac{\Delta f_0^{(T)}(s')\,\mathrm{d}s'}{s'-s}\,, \tag{4}$$

the integration going anticlockwise round the circle. $\Delta f_0^{(T)}(s')$ is the amount

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([11]) S. C. Frautschi: *Phys. Rev. Lett.*, **5**, 159 (1960).
([12]) See also Bowcock *et al.*: ref. ([8]).
([13]) J. S. Ball and D. Y. Wong: *Phys. Rev. Lett.*, **6**, 29 (1961).

by which $f_0^{(T)}(s')$ just inside the circle exceeds the value just outside. Further

$$f_0^{(T)}(s) = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}(\cos\theta)\,\{f_1^{(T)}(q^2,\theta) + \cos\theta f_2^{(T)}(q^2,\theta)\}\,,$$

where f_1, f_2 are given by eq. (7) of I or eq. (3.5), (3.6) of C.G.L.N. ([14]).

Case of c_+. We ignore d-wave π-π interactions. Then for $s' \simeq M^2 \exp[i\varphi']$ where $\varphi' < 45°$, a good approximation is

$$\Delta f_0^{(+)}(s') = \frac{i}{4\pi}\exp[-i\varphi'/2]\int_{1}^{-1} \mathrm{d}(\cos\theta)\ \mathrm{Im}\ A^{(+)}(t)\,,$$

where

$$\mathrm{Im}\ A^{(+)}(t) = \frac{4\pi}{M^2 - t/4}\ \mathrm{Im}\ f_+^0(t)\,, \qquad (t \geqslant 4\mu^2)\,.$$

$f_+^0(t)$ is the helicity amplitude, for angular momentum $J = 0$, for the process $\pi + \pi \to \mathcal{N} + \overline{\mathcal{N}}$. Using the relation

$$\mathrm{d}(\cos\theta) = -\,(1 - \cos\theta)\,\frac{\mathrm{d}t}{t}\,,$$

we get

$$\Delta f_0^{(+)}(s') \simeq \frac{2i}{M^2 t_{\max}} \exp[-i\varphi'/2]\int_{4\mu^2}^{t_{\max}} \mathrm{d}t\ \mathrm{Im}\ f_+^0(t)\,.$$

where

$$t_{\max} = 4\,(\mu^2 \cos^2(\varphi'/2) + M^2 \sin^2(\varphi'/2))\,.$$

Finally, for $q' > 0$, the element of arc $\mathrm{d}s' = iM^2 \exp[i\varphi']\,\mathrm{d}q'$ contributes to the integral (3) an amount

$$\frac{i}{\pi}\exp[i\varphi'/2]\frac{1}{t_{\max}}\int_{4\mu^2}^{t_{\max}} \mathrm{d}t\ \mathrm{Im}\ f_+^0(t)\cdot\frac{\mathrm{d}q'}{s' - s}\,. \tag{5}$$

The equivalent pole for $\Delta^{(+)}(s)$ is at $s_1 = M^2 \exp[i\varphi_1]$, where $\varphi_1 = 33.5°$. Thus from eq. (5) we would expect to find that $\mathrm{Re}\,c_+/\mathrm{Im}\,c_+$ is negative and small. The table shows that this is what we get. For $\Delta_1^{(+)}$, $\mathrm{Re}\,c_+/\mathrm{Im}\,c_+$ is negligible, while for $\Delta_2^{(+)}$, it has the correct sign and contributes to the observed asymmetry of the $\Delta_2^{(+)}$ curve.

We require $\mathrm{Im}\,c_+ > 0$, to fit the $\Delta^{(+)}(s)$ curve. Hence $\int_{4\mu^2}^{\bar{t}} \mathrm{d}t\ \mathrm{Im}\ f_+^0(t) > 0$, where $\bar{t} \simeq 20\,\mu^2$. Using the observed values of $\mathrm{Im}\,c_+$, we can make a rough

([14]) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957).

estimate ([15]) of the average value of $\mathrm{Im}\, f^0_+(t)$ in the range $4\mu^2 \leqslant t \leqslant \bar{t}$. We get $\mathrm{Im}\, \bar{f}^0_+(t) \simeq 70$.

From the order of magnitude of the Born approximation calculation for $\mathrm{Im}\, f^0_+(t)$, in the range $t < 4\mu^2$, it appears that there must be a strong π-π interaction in the $T = 0$ state at energies below $\bar{t} = 20\,\mu^2$. It is not clear whether this must be a resonance.

Case of c_-. The analysis is similar to the c_+ case, but somewhat more complicated. Here $s_1 = M^2 \exp[iq_1]$, where q_1 lies between 26° and 31°. Again the $f_1(q^2, \theta)$ term is predominant for values of q' in the neighbourhood of q_1. Also, in the notation of Frazer and Fulco ([3])

$$\mathrm{Im}\, A^{(-)} + (W - M)\,\mathrm{Im}\, B^{(-)} = 12\pi\{\tfrac{1}{2}t - \mu^2 + (W - M)^2\}\,\mathrm{Im}\, \Gamma_2(t) - \\ - 12\pi(W - M)\,\mathrm{Im}\, \Gamma_1(t)\,.$$

For the values of q' of interest it is a good approximation to retain only the term $-12\pi(W - M)\,\mathrm{Im}\, \Gamma_1(t)$. Then in place of (5) above we have

$$\frac{3M^2}{\pi} \exp[i3q'/4]\,|W_1 - M|\,\frac{1}{t_{\max}} \int_{4\mu^2}^{t_{\max}} \mathrm{d}t\,\mathrm{Im}\, \Gamma_1(t)\,\frac{\mathrm{d}q'}{s' - s}\,, \tag{6}$$

where $s_1 = W_1^2$ and we have replaced $|W' - M|$ by $|W_1 - M|$. If all the important angles q' are not far removed from q_1, then the equivalent residue c_- is nearly real. Further, eq. (6) leads us to expect the small positive values of $\mathrm{Im}\, c_-/\mathrm{Re}\, c_-$ which are obtained from the analysis of the $A^{(-)}$ curves. Using the value $\mathrm{Re}\, c_- = -1.7$ obtained from our analysis, a crude calculation ([15]) gives $\mathrm{Im}\, \bar{\Gamma}_1(t) \simeq -0.08$, where the average is taken over the range $4\mu^2 \leqslant t \leqslant 18\mu^2$. This has the same sign but is much smaller in magnitude than the values derived from the analysis of Frazer and Fulco assuming a $T = 1$ π-π resonance. The discrepancy is again a factor of the order of 10. Thus we can say that at low energies the π-π interaction in the $T = 1$ state is attractive; there appears to be no evidence for a sharp resonance in this state.

6. – Conclusions.

In order to obtain information about the process $\pi + \pi \to \mathcal{N} + \bar{\mathcal{N}}$ we have isolated the contribution from this process to s-wave π-$\mathcal{N}$ scattering. To obtain a sufficiently large range of energy we have extended the calculation into an unphysical region; this can be done in terms of the known experimental s-,

([15]) In this estimate we have assumed that the circle contributes over a 20° arc centred on q_1; also we have taken $\bar{t}$, the average of $t_{\max}$ as q' varies, to be a little greater than $t_{\max}(q_1)$.

p- and d-wave phase shifts for π-$\mathcal{N}$ scattering. The results are shown in Fig. 2. This gives us $A^{(+)}(s)$ and $A^{(-)}(s)$, which represent the contributions from the process $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$ in the $T=0$, $T=1$ states respectively, together with certain short range contributions whose energy dependence is small in the low energy physical region.

The form of $A^{(+)}(s)$ shows that the amplitude for $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$ in the $T=0$ state is large in the region $4\mu^2 < t < 19\mu^2$. This amplitude gives rise to an attractive interaction of fairly long range in low energy π-$\mathcal{N}$ scattering. It is not certain whether a low energy resonance in the $T=0$ state for $\pi+\pi \to \pi+\pi$ is required to explain the magnitude of this amplitude.

The form of $A^{(-)}(s)$ shows that there is an attractive interaction in the $T=1$ state for $\pi+\pi \to \pi+\pi$ at low energies. However the amplitude for $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$ in the $T=1$ state is not large enough to give agreement with the values required by FRAZER and FULCO [3] to explain the nucleon isovector form factor.

The short range interactions in s-wave π-$\mathcal{N}$ scattering appear to be equivalent to a hard core type of repulsion whose radius is 0.2 to 0.3 $\hbar/\mu c$.

A more detailed discussion of several of the points treated in this paper, including a description of the sets of phase shifts we used, will be published later.

* * *

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RIASSUNTO (*)

Inserendo le ampiezze parziali dell'onda s per lo scattering π-$\mathcal{N}$, in una regione non fisica, abbiamo esteso di molto il campo di energia nel quale possiamo esaminare il contributo del processo $\pi+\pi \to \mathcal{N}+\overline{\mathcal{N}}$. Il nostro metodo mette in luce i contributi di questo processo negli stati di spin isotopico $T=0$ e $T=1$. Possiamo valutare le energie $t^{\frac{1}{2}}$ per le quali questo processo è importante. La forma di questi contributi è proprio quella che ci si sarebbe aspettata da considerazioni sugli impulsi angolari. Deduciamo alcune valutazioni delle ampiezze per $\pi+\pi \to \mathcal{N}+\mathcal{N}$. Nel caso $T=0$ l'ampiezza è grande, ma nel caso $T=1$ l'ampiezza è molto minore dei valori predetti dai fattori di forma dell'isovettore nucleonico.

(*) *Traduzione a cura della Redazione.*

Spin Precession in Classical Relativistic Mechanics.

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(ricevuto il 13 Febbraio 1961)

Summary. — The motion of a point-particle with spin according to relativistic classical mechanics is shown to possess the following properties. A free particle is not required to move along a straight line, but may move along a helix in space, or along a hyperbola in space-time. The helical motion is the classical analogue of zitterbewegung and the hyperbolic motion is the classical analogue of pair production. The effective mass M of the particle is in general not equal to its rest mass. In a magnetic field B, the particle automatically possesses a magnetic moment $eB\sigma/Mc$, where σ is the spin. The effective mass is a constant of the motion in a uniform electric or magnetic field.

1. – Equations of motion.

If it is assumed that the polar vector $\boldsymbol{\tau} = i(\sigma_{41}, \sigma_{42}, \sigma_{43})$, associated with the spin $\boldsymbol{\sigma} = (\sigma_{23}, \sigma_{31}, \sigma_{12})$ of a point-particle, vanishes in the rest system of the particle:

$$\sigma_{ij} v_j = 0 \tag{1}$$

and that the moment of inertia of the particle about any axis perpendicular to the spin vanishes, and if radiation reaction terms are neglected, the classical equations of motion of the particle are [1]

$$\dot{\sigma}_{ij} = (\sigma_{im} v_j - \sigma_{jm} v_i) \left(\dot{v}_m - \frac{eg}{2mc^2} f_{mk} v_k \right) + \frac{eg}{2mc^2} (f_{ik}\sigma_{kj} - \sigma_{ik} f_{kj}) , \tag{2}$$

$$P_i = mcv_i - \dot{\sigma}_{ij} v_j - \frac{eg}{4mc^2} v_i \sigma_{kl} f_{kl} - \frac{eg}{2mc^2} \sigma_{ij} f_{jk} v_k , \tag{3}$$

$$\dot{P}_i = \frac{e}{c} f_{ik} v_k + \frac{eg}{4mc^2} \sigma_{kl} \partial_i f_{kl} . \tag{4}$$

[1] H. J. Bhabha and H. C. Corben: *Proc. Roy. Soc.*, A **178**, 273 (1941); eq. (103) with $K = 0$, $IcS_{ij} = \sigma_{ij}$, $g_2 = I(eg/2mc^2)$. Cf. J. Weysenhoff: *Acta Phys. Polon.*, **9**, 7 (1947).

Here

$$v_i = \dot{x}_i \equiv \frac{\mathrm{d}x_i}{\mathrm{d}s}, \qquad x = ict, \qquad \frac{\mathrm{d}}{\mathrm{d}s} = \frac{\gamma}{c}\frac{\mathrm{d}}{\mathrm{d}t}, \qquad \gamma = (1-\beta^2)^{-\frac{1}{2}},$$

and g is an arbitrary parameter relating the explicit magnetic and electric moment tensor with the spin tensor

$$\mu_{ij} = \frac{eg}{2mc}\sigma_{ij}. \tag{5}$$

Since for a particle at rest eq. (2) for the axial components of σ_{ij} reduces to

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} = \frac{eg}{2mc}(\boldsymbol{\sigma}\times\boldsymbol{B}), \tag{6}$$

it is commonly assumed that in order to introduce into the theory the correct gyromagnetic ratio for the electron or μ-meson (apart from radiative corrections) it is necessary to set $g = 2$.

However, it was shown in reference [1] that to describe the electron and μ-meson classically, it is necessary to put $g = 0$, since if g is chosen to be equal to 2, the scattering of light by this dipole is quite unrelated to the Klein-Nishina formula for frequencies for which classical theory should be applicable. If, however, we set $g = 0$ in eq. (6), how can we describe spin precession in a magnetic field in a manner which corresponds to observation?

It is the purpose of this paper to point out that if we set $g = 0$, *i.e.*, if no explicit magnetic moment is ascribed to the particle, the classical equations of motion automatically lead to the correct gyromagnetic ratio apart from anomalous terms, for they permit even a free spinning point-particle to move in a helix, and to possess a magnetic moment even if none has been ascribed to it. The parameter g is therefore proportional to the *anomalous* part of the magnetic moment.

For $g = 0$, eq. (2), (3), and (4) become

$$\left.\begin{aligned} \dot{\sigma}_{ij} &= (\sigma_{im}v_j - \sigma_{jm}v_i)\dot{v}_m = (v_jP_i - v_iP_j), \\ P_i &= mcv_i + \sigma_{ij}\dot{v}_j, \\ \dot{P}_i &= \frac{e}{c}f_{ik}v_k. \end{aligned}\right\} \tag{7}$$

We note that if eq. (1) is satisfied at any instant, then the equations of motion are such that it continues to be satisfied.

Writing $P_i = (\boldsymbol{P}, iW/c)$, we have

$$(7^1)\qquad \begin{cases} m\boldsymbol{v}\gamma - \dfrac{\gamma^2}{c^2}\,\boldsymbol{\sigma}\times\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{P}\,, \\ mc^2\gamma - \dfrac{\gamma^2}{c^2}\,\boldsymbol{v}\cdot\boldsymbol{\sigma}\times\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = W\,, \\ \dfrac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} = \boldsymbol{P}\times\boldsymbol{v}\,, \end{cases}$$

since from eq. (1) $\boldsymbol{\tau} = \boldsymbol{v}\times\boldsymbol{\sigma}/c$, $\mathrm{d}\boldsymbol{\tau}/\mathrm{d}t = e\boldsymbol{P} - (W/c)\boldsymbol{v}$. It also follows from the equations of motion that

$$(8)\qquad v_i P_i = -mc$$

or

$$(8^1)\qquad \gamma(W - \boldsymbol{v}\cdot\boldsymbol{P}) = mc^2\,.$$

For $g = 0$, the constant parameters, in terms of which the particle is described, are its rest mass m, its charge e, and the magnitude σ_0 of the spin of the particle in its rest system:

$$(9)\qquad \sigma_0^2 \equiv \frac{1}{2}\,\sigma_{ij}\sigma_{ij} = \sigma^2 - \tau^2 = \frac{\sigma^2}{\gamma^2} + \frac{(\boldsymbol{v}\cdot\boldsymbol{\sigma})^2}{c^2}\,.$$

It also follows from the equations of motion that if we write

$$(10)\qquad P_i P_i - \frac{e}{c}\,\sigma_{ik} f_{ik} = -M^2c^2\,,$$

then

$$(11)\qquad \frac{\mathrm{d}M}{\mathrm{d}s} = \frac{e}{2Mc^3}\,\sigma_{ik}\frac{\mathrm{d}f_{ik}}{\mathrm{d}s} = \frac{e}{Mc^3}\left(\boldsymbol{\sigma}\cdot\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}s} + \boldsymbol{\tau}\cdot\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}s}\right).$$

Thus in a uniform static field M is a constant of the motion, although in general it is not equal to the rest mass m. If we write $W = E - e\varphi$, $\boldsymbol{P} = \boldsymbol{p} - (e/c)\boldsymbol{A}$, eq. (10) becomes

$$(E - e\varphi)^2 = \left(p - \frac{e}{c}\,A\right)^2 + M^2c^4 - 2ec(\boldsymbol{\sigma}\cdot\boldsymbol{B} + \boldsymbol{\tau}\cdot\boldsymbol{E})\,,$$

or, for $Mc \gg |\boldsymbol{p} - (e/c)\boldsymbol{A}|$ and sufficiently small fields

$$(11^1)\qquad E \doteq Mc^2 + e\varphi + \frac{1}{2M}\left(p - \frac{e}{c}\,A\right)^2 - \frac{e}{Mc}\,(\boldsymbol{\sigma}\cdot\boldsymbol{B} + \boldsymbol{\tau}\cdot\boldsymbol{E})\,.$$

Thus in a uniform field, the particle behaves as if it had a mass M, a magnetic moment $e\boldsymbol{\sigma}/Mc$, and an electric moment $e(\boldsymbol{v}\times\boldsymbol{\sigma})/Mc^2$ even although no explicit magnetic moment has been ascribed to it.

It follows from the equations of motion (7) that

$$\dot{v}_j P_i = 0 \tag{12}$$

so that

$$\dot{\sigma}_{ij}\dot{v}_j = -v_i\dot{v}_j P_j = 0\,.$$

The equations of motion may therefore be written thus

$$mc\dot{v}_i = \left(\frac{e}{c}f_{ik} + \ddot{\sigma}_{ik}\right)v_k\,, \tag{13}$$

$$\dot{\sigma}_{ij} = (v_j\sigma_{ik} - v_i\sigma_{jk})\,\dot{v}_k\,, \tag{14}$$

according to which the particle moves as if under the influence of the effective field

$$F_{ik} = f_{ik} + \frac{c}{e}\,\ddot{\sigma}_{ik}\,. \tag{15}$$

In the absence of any external field, it follows that a free particle is not required to move in a straight line.

It is convenient to introduce the four-vector $\boldsymbol{\rho}_i$ such that

$$\dot{\boldsymbol{\rho}}_i = \boldsymbol{v}_i - \frac{\boldsymbol{P}_i}{Mc}\,,$$

and to define an effective spin tensor S_{ij} by

$$S_{ij} = \sigma_{ij} + (\varrho_i P_j - \varrho_j P_i)\,. \tag{16}$$

It follows from (7) that

$$\dot{S}_{ij} = \frac{e}{c}\,(\varrho_i f_{jk} - \varrho_j f_{ik})\,v_k\,, \tag{17}$$

so that S_{ij} is a constant of the motion for a free particle. Since

$$v_i P_j = v_j P_i + \dot{\varrho}_i P_j - \dot{\varrho}_j P_i\,,$$

we may write, for $i = 1, 2, 3,\ j = 4$

$$\boldsymbol{v} = c^2\frac{\boldsymbol{P}}{W} + \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} + \frac{ic}{W}\frac{\mathrm{d}\varrho_4}{\mathrm{d}t}\boldsymbol{P}\,. \tag{18}$$

2. – Motion of a free particle.

In the absence of any fields, the four-vector P_i is constant and

$$W^2 = P^2c^2 + M^2c^4 .$$

For the special case $\boldsymbol{P} = 0$, eq. (7[1]) has a solution

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} , \qquad \boldsymbol{\Omega} = - \frac{Mc^2}{\sigma^2} \boldsymbol{\sigma} , \qquad \boldsymbol{\sigma} = \text{const} , \tag{19}$$

in which the particle moves in a circle normal to σ and of radius

$$r = \beta \frac{\sigma}{Mc} , \tag{20}$$

with

$$W = Mc^2 = \frac{mc^2}{\gamma} = - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega} . \tag{21}$$

Because of the intrinsic spin, the center of mass of the particle is not at the position of the particle, remaining instead fixed at the center of the circle around which the particle is moving. In this sense, the classical equations reflect the zitterbewegung of the Dirac equation and reveal the physical origin of the magnetic moment. From (15), this free particle motion takes place as if the particle were bound to the center of the circle by an effective electric field directed towards the center and of magnitude given by

$$eE_{\text{eff}} = \frac{\gamma^2}{c^2} \Omega^2 \sigma v .$$

A more general solution of eq. (7[1]) for a free particle, with $\boldsymbol{P} \neq 0$, is (cf. (18), with $\varrho_4 = \text{const}$)

$$\boldsymbol{r}(t) = \frac{c^2 t}{W} \boldsymbol{P} + \boldsymbol{\rho}(t) + \boldsymbol{R}_0 , \tag{22}$$

where

$$\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{\rho} , \qquad \boldsymbol{\Omega} = \frac{- M^2c^4}{W(\boldsymbol{J} \cdot \boldsymbol{P})} \boldsymbol{P} , \tag{23}$$

$$\boldsymbol{R}_0 = \frac{\boldsymbol{P} \times \boldsymbol{J}}{P^2} = \text{const} , \tag{24}$$

$$\boldsymbol{J} = \boldsymbol{\sigma} + \boldsymbol{r} \times \boldsymbol{P} = \text{const}, \tag{25}$$

$$M^2 = \left(\frac{m\sigma_0 P}{\boldsymbol{J} \cdot \boldsymbol{P}} \right)^2, \tag{26}$$

$$\gamma = \frac{mW}{M^2 c^2}. \tag{27}$$

Since $\mathrm{d}\boldsymbol{\rho}/\mathrm{d}t$ is at right angles to $\boldsymbol{P}$, it is here assumed that $\boldsymbol{\rho} \cdot \boldsymbol{P} = 0$, since any constant component of $\boldsymbol{\rho}$ in the direction of $\boldsymbol{P}$ may be incorporated into the first term on the right hand side of (22) by changing the time origin.

In verifying that (22) is indeed a solution of (7¹), we first note that the last of eq. (7¹) yields the total angular momentum integral (25). It follows that $\boldsymbol{\sigma}$ and $\boldsymbol{r} \times \boldsymbol{P}$ are not separately constants of the motion, but substitution of (22) into (25) would yield, using (24),

$$\boldsymbol{\sigma} = - \boldsymbol{\rho} \times \boldsymbol{P} + \frac{\boldsymbol{P} \cdot \boldsymbol{J}}{P^2} \boldsymbol{P}, \tag{28}$$

and since from (22)

$$\boldsymbol{v} = \frac{c^2 \boldsymbol{P}}{W} + \boldsymbol{\Omega} \times \boldsymbol{\rho}, \tag{29}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = - \Omega \, \boldsymbol{\rho}, \tag{30}$$

it is readily shown that the remaining eq. (7¹) are satisfied provided that (23), (27) are valid and that

$$\varrho_0 = |\varrho| = \left| \frac{\boldsymbol{J} \cdot \boldsymbol{P}}{MPc} \right| \left(1 - \frac{M^2}{m^2} \right)^{\frac{1}{2}}. \tag{31}$$

Since, however, from (29)

$$v^2 = \frac{c^4 P^2}{W^2} + \Omega^2 \varrho^2,$$

it follows from (27) that $|\varrho|$ is in fact given by (31).

We note also from (28), (29) that

$$\frac{\boldsymbol{v} \cdot \boldsymbol{\sigma}}{c} = \frac{c}{W} \boldsymbol{J} \cdot \boldsymbol{P} + \varrho^2 \boldsymbol{\Omega} \cdot \boldsymbol{P} = \frac{M^2 c}{m^2 W} \boldsymbol{J} \cdot \boldsymbol{P},$$

by (23), (31). Hence, from (9), (27), we obtain (26).

The sum of the intrinsic spin and the angular momentum due to the helical motion

$$\boldsymbol{S} = \boldsymbol{\sigma} + \boldsymbol{\rho} \times \boldsymbol{P} \tag{32}$$

may be defined as the effective spin of the particle (cf. eq. (16)). From eq. (28), it follows that

$$\boldsymbol{S} = \frac{P \cdot J}{P^2} \boldsymbol{P}, \tag{33}$$

i.e., that $\boldsymbol{S}$ is a constant of the motion directed either parallel or antiparallel to $\boldsymbol{P}$. The total angular momentum $\boldsymbol{J}$ may then be broken into two parts which are separately constants of the motion for a free particle:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S},$$

where $\boldsymbol{L} = \boldsymbol{R}_0 \times \boldsymbol{P}$, $\boldsymbol{R}_0$ being the vector (24) drawn from the origin perpendicular to the axis of the helix along which the particle is moving. During one revolution of ϱ, the particle moves forward a distance $\lambda_0 = 2\pi\lambda\!\!\!^{-}_0$ where

$$\lambda\!\!\!^{-}_0 = \frac{c^2 P}{W\Omega} = \frac{\boldsymbol{P}\cdot\boldsymbol{J}}{M^2c^2} = \frac{SP}{M^2c^2}. \tag{34}$$

Thus $\lambda\!\!\!^{-}_0 \lambda\!\!\!^{-}_d = S\hbar/M^2c^2$, where $\lambda\!\!\!^{-}_d = \hbar/P$ is the de Broglie wave length.

Since

$$M^2c^2 = -P_i P_i = M^2c^2 - \sigma_{ij}\dot{v}_j \sigma_{ik}\dot{v}_k$$

or

$$m^2 - M^2 = \frac{\gamma^4}{c^6}\left[\left(\boldsymbol{\sigma}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)^2 - \left(\frac{\boldsymbol{v}\times\boldsymbol{\sigma}}{c}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)^2\right], \tag{35}$$

it follows that $|M|$ may be greater or less than m. It might have been expected that the effective mass would be greater than the rest mass, but a point particle with spin cannot be thought of as the limit of a distribution of matter with positive energy density. The energy associated with the internal motion is therefore not necessarily positive. For the motion described by (22) $|M| < m$ so that ϱ_0, given by (31), is real and may be written in the form

$$\varrho_0 = \left|\frac{S}{Mc}\right|\left(1 - \frac{M^2}{m^2}\right)^{\frac{1}{2}}.$$

Since up to this point, only the square of M has appeared in the analysis, the sign of the effective mass M is ambiguous. We may therefore define M

to be given from (26) by

$$M = + \frac{m\sigma_0 P}{\boldsymbol{J}\cdot\boldsymbol{P}}, \tag{36}$$

and take the parameters m, σ_0 which denote intrinsic properties of the particle to be positive. It then follows that

$$M\boldsymbol{S} = m\sigma_0 \frac{\boldsymbol{P}}{P}, \tag{37}$$

so that the sign of M is determined by the sign of $\boldsymbol{S}\cdot\boldsymbol{P}$.

From the point of view of the Dirac equation, zitterbewegung and pair production may be regarded as different aspects of the same phenomenon (2). It is therefore not surprising that one may find formal solutions of the equations of motion of a free spinning particle according to classical relativistic mechanics which also correspond to pair production. By analogy with (19), we look for solutions of (7¹) for which

$$\sigma_x = 0\,, \qquad \sigma_y = 0\,, \qquad v_z = 0\,.$$

However, instead of allowing W to be the only non-vanishing component of the constant four-vector P_i, we suppose that P_x is the only non-vanishing component:

$$P_y = 0\,, \qquad P_z = 0\,, \qquad W = 0\,.$$

Hence

$$P_x = iMc$$

so that either P_x or M is imaginary. The formal solutions are then given by

$$\left|\begin{aligned} y &= a(e^{\theta} - \eta e^{-\theta})\,,\\ t &= \frac{a}{c}(e^{\theta} + \eta e^{-\theta})\,,\\ \gamma &= \frac{iMc}{\sigma_0}\, y\,,\\ v_x &= \frac{m\sigma_0}{M^2 y}\,, \end{aligned}\right. \tag{38}$$

where $\eta = \pm 1$ and

$$\eta a^2 = \left(\frac{\sigma_0}{2Mc}\right)^2 \left(1 - \frac{m^2}{M^2}\right). \tag{39}$$

(2) H. C. Corben: *Physics Today*, **7**, no. 3, 10 (1954).

Since a is real and we assume σ_0 and m to be real, the sign of η is determined as follows

$$\eta = -1 \qquad M < m \text{ or } M \text{ imaginary},$$
$$\eta = +1 \qquad M > m .$$

Hence, for $M < m$ or M imaginary

$$\left| \begin{aligned} y &= \frac{\sigma_0}{M^2 c}(m^2 - M^2)^{\frac{1}{2}} \cosh\theta , \\ t &= \frac{\sigma_0}{M^2 c^2}(m^2 - M^2)^{\frac{1}{2}} \sinh\theta , \\ \gamma &= \frac{i}{M}(m^2 - M^2)^{\frac{1}{2}} \cosh\theta , \\ x &= \frac{m\sigma_0}{M^2 c}\theta ; \end{aligned} \right. \tag{41}$$

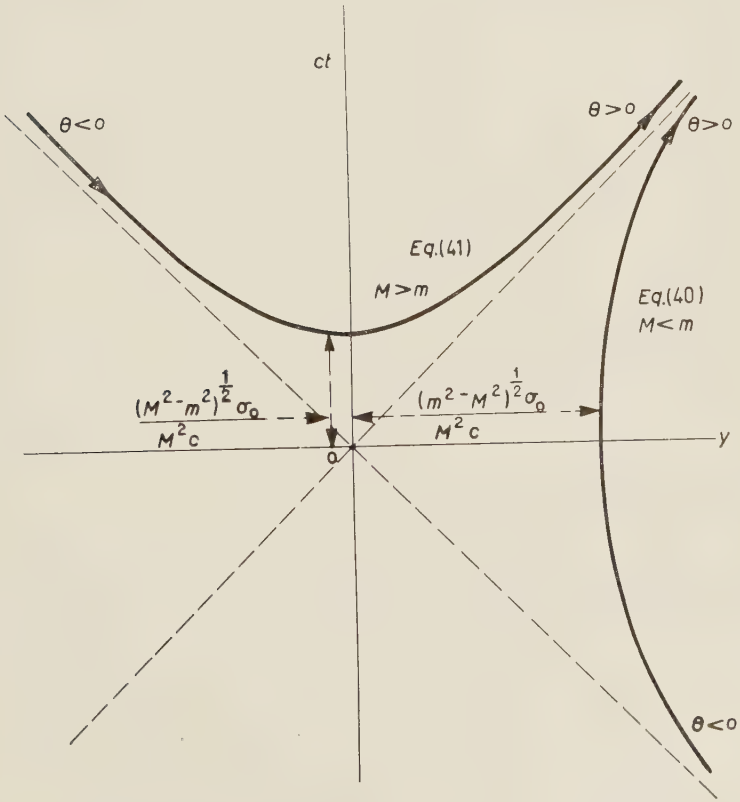

Fig. 1. – Projection in y-t plane of solutions (40), (41) for a free particle with M real, spin in z direction. For given m, the values of M belonging to the two branches are not equal. In these solutions, the velocity always exceeds c.

whereas for $M > m$

$$(41)\qquad \begin{cases} y = \dfrac{\sigma_0}{M^2c^2}\,(M^2 - m^2)^{\frac{1}{2}} \sinh\theta\,, \\ t = \dfrac{\sigma_0}{M^2c^2}\,(M^2 - m^2)^{\frac{1}{2}} \cosh\theta\,, \\ \gamma = \dfrac{i}{M}\,(M^2 - m^2) \sinh\theta\,, \\ x = \dfrac{m\sigma_0\theta}{M^2c}\,. \end{cases}$$

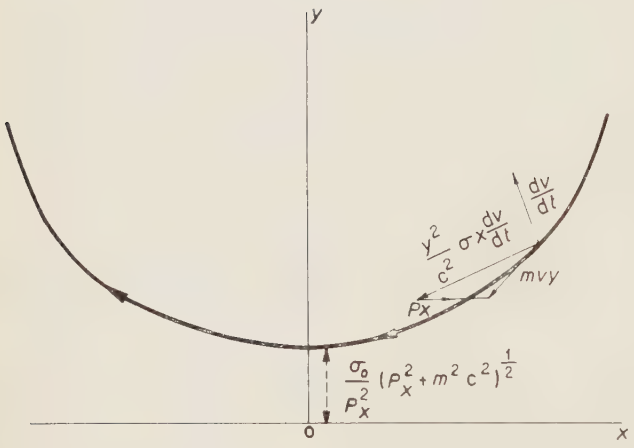

Fig. 2. – Self-accelerated solution for a free particle moving in x-y plane, with spin in z direction. P_x real, $v < c$, $W = 0$, $J = 0$.

In either case, the projection of the space-time path on the y-t plane is a hyperbola. For imaginary M, γ is real and the velocity of the particle is less than c. For real M, imaginary P_x, $v > c$. These solutions are illustrated in Fig. 1 and 2. In each case

$$\boldsymbol{J} = \boldsymbol{\sigma} + \boldsymbol{r} \times \boldsymbol{P} = 0$$

and the total energy W also vanishes.

3. – Motion in uniform external fields.

In the presence of a uniform magnetic field B in the z direction, solution (19) for a free particle is modified as follows

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2\,,$$

$$\boldsymbol{\sigma} = \sigma\boldsymbol{e}_3\,, \qquad \boldsymbol{B} = B\boldsymbol{e}_3\,,$$

$$v_1 = u \sin \Omega t + w \cos \Omega t\,,$$

$$v_2 = -\,w \sin \Omega t + u \cos \Omega t\,,$$

where

$$\Omega^2 - \frac{mc^2}{\gamma\sigma}\Omega + \frac{eBc}{\gamma^2\sigma} = 0 ,$$

or

$$(42) \qquad \begin{cases} \Omega = \dfrac{mc^2}{2\gamma\sigma}\left[1 \pm \left(1 - \dfrac{4eB\sigma}{m^2c^3}\right)^{\frac{1}{2}}\right], \\ \quad \doteqdot \dfrac{mc^2}{\gamma\sigma} - \dfrac{eB}{m\gamma c} \qquad \text{or} \qquad \dfrac{eB}{m\gamma c}, \end{cases}$$

for $eB\sigma \ll m^2c^3$. The second of these solutions represents motion in a large circle of radius $\sim (eB)^{-1}$, the usual motion when spin can be neglected. The former solution represents a modulation of the frequency of the zitterbewegung by the external magnetic field. However, we should have expected this modulation to be $eB/Mc \doteqdot eB\gamma/mc$, since $M = m/\gamma$ is the effective mass. To see how this arises, we note that, when the magnetic field is turned on, γ and σ are changed. The angular momentum is now

$$\boldsymbol{J} = \boldsymbol{\sigma} + \boldsymbol{r} \times \boldsymbol{p}$$

where

$$\boldsymbol{p} = \boldsymbol{P} + \frac{e}{c}\boldsymbol{A} .$$

Since

$$\boldsymbol{P} = \frac{eB}{c}\left[\boldsymbol{e}_1 x_2 - \boldsymbol{e}_2 x_1\right],$$

$$\boldsymbol{A} = \tfrac{1}{2}B\left[-\boldsymbol{e}_1 x_2 + \boldsymbol{e}_2 x_1\right],$$

the angular momentum is

$$\boldsymbol{e}_3\left[\sigma - \frac{eBr^2}{2c}\right],$$

where $\sigma = \gamma\sigma_0$. If γ_0 denotes the value of γ in the absence of the field, the angular momentum is unchanged by the field if

$$\gamma_0 = \gamma - \frac{eBr^2}{2c\sigma_0},$$

so that

$$\delta\gamma = \frac{eBr^2}{2c\sigma_0} .$$

Hence from (42), the change in Ω when the field is turned on is

$$\delta\Omega = -\frac{mc^2}{\gamma^2\sigma^2}\,\delta(\gamma\sigma) - \frac{eB}{m\gamma c}\,,$$

$$= -\frac{2mc^2}{\gamma^2\sigma}\,\delta\gamma - \frac{eB}{m\gamma c} = -\frac{eB\gamma}{mc} = -\frac{eB}{Mc}\,,$$

using (20). Thus when a magnetic field is turned on parallel to the spin, the angular frequency of precession decreases by eB/Mc, provided that the total angular momentum of the particle about the fixed center stays constant.

Approximate solutions of the equations of motion in an inhomogeneous field may be obtained by adapting the method introduced by Alfvén [3] for studying the motion of the guiding center. This method may also be adapted to the case of motion in a uniform electric field E. If E and the initial momentum P_0 are in the z direction, we have

$$\boldsymbol{P} = (P_0 + eEt)\boldsymbol{e}_3\,.$$

The solution which corresponds to (22) for $E = 0$, correct to first order in eE is

$$\text{(43)}\qquad \begin{cases} v = \dfrac{\varepsilon}{\gamma}[\boldsymbol{e}_1\cos\Omega t + \boldsymbol{e}_2\sin\Omega t] + u(t)\boldsymbol{e}_3\,, \\ \sigma = K\gamma[e_1\cos\Omega t + \boldsymbol{e}_2\sin(\Omega t + \delta)] + \alpha\boldsymbol{e}_3\,, \end{cases}$$

where ε, K, α are constants and

$$\text{(44)}\qquad \begin{cases} \boldsymbol{P} = \dfrac{Mu}{(1-u^2/c^2)^{\frac{1}{2}}}\boldsymbol{e}_3\,, \qquad W = \dfrac{Mc^2}{(1-u^2/c^2)^{\frac{1}{2}}}\,, \\ M = \dfrac{m}{(1+\varepsilon^2/c^2)^{\frac{1}{2}}}\,, \qquad \gamma = \dfrac{(1+\varepsilon^2/c^2)^{\frac{1}{2}}}{(1-u^2/c^2)^{\frac{1}{2}}} = \dfrac{mW}{M^2c^2}\,, \\ K\gamma \doteq -\dfrac{\alpha\varepsilon}{mc^2}P\,, \\ \delta \doteq \sin\delta = -\dfrac{\alpha euE}{M^2c^4}\,, \\ \Omega = -\dfrac{mc^2}{\gamma\alpha}\,. \end{cases}$$

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[3] H. Alfvén: *Cosmical Electrodynamics* (Oxford, 1950).

The approximation made here is equivalent to assuming that $|\delta| \ll 1$. It then follows that

$$\left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{eE}{M},$$

so that, since the acceleration is longitudinal, M is in fact the effective rest mass. The radius of the helical motion is still given by eq. (31), the component of $\boldsymbol{\sigma}$ in the direction of $\boldsymbol{P}$ remaining constant.

RIASSUNTO (*)

Dimostriamo che secondo la meccanica relativistica classica il moto di una particella puntiforme con spin, possiede le seguenti proprietà. Non è necessario che una particella libera si muova lungo una retta, ma si può muovere nello spazio lungo un'elica, o lungo un'iperbole nello spazio tempo. Il moto elicoidale è l'analogo classico della zitterbewegung e il moto iperbolico è l'analogo classico della produzione di coppie. La massa effettiva M della particella in generale non è uguale alla sua massa di riposo. In un campo magnetico B, la particella possiede automaticamente un momento magnetico $eB\sigma/Mc$, in cui σ è lo spin. La massa effettiva è una costante del moto in un campo magnetico od elettrico uniforme.

(*) *Traduzione a cura della Redazione.*

Radiative Corrections to Pion Production in e^+-e^- Collisions.

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(ricevuto il 16 Febbraio 1961)

Summary. — Radiative corrections to the processes e^+-$e^- \to n$ pions are evaluated in relation to the planned colliding beam experiments. A theorem is proved which shows the possibility of separating experimentally the contribution of the γ-n pion vertex from the contribution of the 2γ-n pion vertex.

In a recent work Cabibbo and Gatto [1] have discussed the possibility of direct measurement of the form factors of the photon-pion vertex through processes of the type:

$$e^+ + e^- \to n \text{ pions} . \tag{1}$$

They have also obtained in the first electromagnetic approximation the expression of the corresponding cross-section in the center-of-mass system. We will here examine in which way their results can be modified if the radiative corrections in the second electromagnetic approximations are taken into account. The general situation is then characterized by Feynman's diagrams of Fig. 1, where F and G represent the « complete » vertices γ-$n\pi$ and 2γ-$n\pi$; that is they correspond to the sum of all the diagrams with external lines of this type.

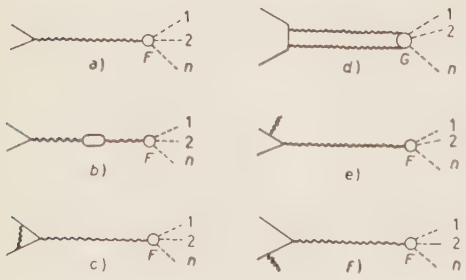

Fig. 1.

Experiments on processes of type (1) should precisely measure F and G.

[1] N. Cabibbo and R. Gatto: *Phys. Rev. Lett.*, **4**, 313 (1960).

We have neglected the diagrams corresponding to emission of photons from the second vertex. Taking into account the difference of mass between pions and electrons, their contribution is probably negligible in comparison with the contribution from the diagrams (1*e*) and (1*f*).

The corrections corresponding to diagrams 1*b*), 1*c*), 1*e*), 1*f*) do not involve new photon-pion vertices and therefore they will be later calculated with the usual techniques.

The correction corresponding to the diagram with exchange of two virtual photons (Fig. 1*d*) contains the new unknown vertex, 2 γ-pions.

On the basis of general considerations we will show that it is easy to isolate experimentally the contributions to the cross-sections from the graphs with exchange of two photons; *i.e.* it is possible to distinguish the contributions of the two form factors. Our proof is based on three points:

a) In processes of type (1) the initial state, being eigenstate of the charge with eigenvalue 0, is a superposition of two eigenstates of the charge-parity operator C relative to eigenvalues $+1$ and -1:

$$|i\rangle = |i^+\rangle + |i^-\rangle \quad \text{with} \quad C|i^\pm\rangle = \pm|i^\pm\rangle . \tag{2}$$

b) Calling S' and S'' the terms of the operator S corresponding to diagrams of Fig. 1*a* and 1*d*) respectively we have

$$S'|i^+\rangle = 0 , \qquad S''|i^-\rangle = 0 . \tag{3}$$

In fact for instance

$$S'|i^+\rangle = (ie)^2 \int \mathrm{d}x_1 \, \mathrm{d}x_2 \, i\Delta_f(x_1, x_2) J_\pi(x_1)|0\rangle \langle 0|J_e(x_2)|i^+\rangle = 0 ,$$

since

$$\langle 0|J_e(x_2)|i^+\rangle = \langle 0C^{-1}|C J_e(x_2) C^{-1}|Ci^+\rangle = -\langle 0|J_e(x_2)|i^+\rangle .$$

In the same way it is shown that $S''|i^-\rangle = 0$.

c) We will now show that if the set of final states F distinguished from the measurement is an invariant subspace for the operator C, the contribution to the cross-section of the interference term between the matrix elements corresponding to the diagrams of Fg. 1*a*) and 1*d*) is zero. In fact in this case we can assume as basis in F a set of vectors $|f_s\rangle$ wich are eigenstates of C.

$$C|f_s\rangle = \pm|f_s\rangle . \tag{5}$$

The contribution to the transition probability in e^6 from the initial state $|i\rangle$

to the set of final states F, due to the diagrams with exchange of two photons is

$$
(6)\qquad
\begin{cases}
2\,\mathrm{Re}\{\sum_s \langle i|S'|f_s\rangle\langle f_s|S''|i\rangle\} = \\
2\,\mathrm{Re}\{\sum_s \langle i^-|S'|f_s\rangle\langle f_s|S''|i^+\rangle\} = \\
2\,\mathrm{Re}\{\sum_s \langle i^-C^{-1}|CS'C^{-1}|Cf_s\rangle\langle f_sC^{-1}|CS''C^{-1}|Ci^+\rangle\} = \\
-2\,\mathrm{Re}\{\sum_s \langle iS'|f_s\rangle\langle f_s|S''|i\rangle\} = 0\,.
\end{cases}
$$

Let us now examine to wich experimental situation does the condition that F be an invariant subspace of C correspond. It is evidently sufficient that the apparatus revealing the final particles do not distinguish the π^+ from the π^-. This condition can certainly be verified in the experiments with intersecting beams in project at Stanford and at Frascati.

For the simplest processes of type (1), that is $e^+ + e^- \to \pi^+ + \pi^-$, it follows that, apart from particular experimental situations, the term of the cross-section in e^6 due to the diagram d) is an odd function of $\cos\theta$ (where θ is the angle between the momenta of the electron and of the π^-). Therefore it does not contribute to the differential cross-section for $\theta = 90^\circ$ and to the total cross-section. As instead the contribution of the other diagrams with exchange of one photon are even functions of $\cos\theta$, it is possible to separate the two types of contributions in an experience with variable θ: this means that it is possible to measure separately the form factors for the vertices 2π-γ and 2π-2γ. The above conclusion is valid also for similar processes like $e^- + e^- \to \mu^+ + \mu^-$, $e^+ + e^- \to K^+ + K^-$, etc. Thus in an experiment $e^+ + e^- \to \mu^+ + \mu^-$ it is possible to distinguish the contribution to the cross-section in e^6 due to the diagrams wich are unimportant for a check of renormalization theory, from the contributions of vertex and self-energy corrections.

Keeping in mind the above result, we have calculated the expression of the cross-section to be used if one wants to measure the form factor of the « complete » vertex γ-$n\pi$ through an experience of type (1).

The formula is valid for the conditions stated above, *i.e.* no contribution from the diagram with vertex 2γ-$n\pi$; we have taken into account the fact that the electrons are certainly relativistic, and we have assumed that the maximum energy ε of the bremsstrahlung photons (see Fig. 1*e* and 1*f*) is small with reference to the energy of the emitting particles.

In this hypothesis the expression of the correction is independent of the specific form of vertex γ-$n\pi$.

Using the usual techniques to calculate radiative corrections we find

$$
(7)\qquad \mathrm{d}\sigma_n^{(1)} = \mathrm{d}\sigma_n^{(0)}(1 + \delta_{SE} + \delta_V + \delta_B)\,,
$$

where $d\sigma_n^{(0)}$ and $d\sigma_n^{(1)}$ are the cross-sections in the first and in the second electromagnetic approximation of process (1), while δ_{SE}, δ_V and δ_B are the percentage corrections due to diagrams of Fig. 1*b*), 1*c*), 1*e*) and 1*f*). Expressions for $d\sigma_n^{(0)}$ are given in (1); for the corrections we have obtained

$$(8) \qquad \begin{cases} \delta_{SE} = \dfrac{2\alpha}{\pi}\dfrac{2}{3}\left\{\ln\dfrac{2E}{m} - \dfrac{5}{6}\right\}, \\ \delta_V = -\dfrac{2\alpha}{\pi}\left\{\left(1 - 2\ln\dfrac{2E}{m}\right)\ln\dfrac{\lambda}{m} + \left(\ln\dfrac{2E}{m}\right)^2 - \dfrac{3}{2}\ln\dfrac{2E}{m} + 1 + \dfrac{\pi^2}{6}\right\}, \end{cases}$$

$$(9) \qquad \delta_B = -\frac{2\alpha}{\pi}\left\{\left(1 - 2\ln\frac{2E}{m}\right)\ln\frac{2\varepsilon}{\lambda} + \left(\ln\frac{2E}{m}\right)^2 - \ln\frac{2E}{m} + \frac{\pi^2}{6}\right\},$$

where α is the fine structure constant, E is the energy and m the mass of the electron (or positron), ε is the maximum energy of the bremsstrahlung photons, λ is the fictitious mass of the photon, that disappears in the expression for global correction.

It is to be noted that (7) and (8) are also valid for $e^+ + e^- \to \mu^+ + \mu^-$, if $d\sigma^{(0)}$ and $d\sigma^{(1)}$ are the corresponding cross-sections in e^4 and e^6, neglecting radiative corrections due to the μ-mesons. To take them into account it is sufficient to replace in (7) $(1 + \delta_{SE} + \delta_V + \delta_B)$ with $(1 + \delta_{SE} + \delta_V + \delta_B + \delta_{SE}^{(\mu)} + \delta_V^{(\mu)} + \delta_B^{(\mu)})$, where the $\delta^{(\mu)}$'s are obtained from the δ's by substituting the electron's mass with the μ-meson's one.

In this formula the effects due to creation by the intermediate photon of virtual particles heavier than electrons have been neglected; a most important contribution could come from the two-pion intermediate states ([2,3]).

In agreement with ([3]) they are negligible.

* * *

We are indebted to Professor B. Touschek for helpful assistance and encouragement.

([2]) L. M. Brown and F. Calogero: *Phys. Rev. Lett.*, **4**, 315 (1960).
([3]) Yung-Su-Tsai: *Phys. Rev.*, **120**, 269 (1960).

RIASSUNTO

Sono state calcolate le correzioni radiative ai processi $e^+ + e^- \to n$ pioni in relazione agli esperimenti a fasci incrociati in progetto. Viene dimostrato un teorema che indica la possibilità di separare sperimentalmente il contributo del vertice γ-n pioni dal contributo del vertice 2γ-n pioni.

K-Meson-Nucleon Scattering.

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(ricevuto il 24 Febbraio 1961)

Summary. — Assuming that the isotropy of K$\mathcal{N}$-scattering in $I=1$ is due to the cancellation of p-wave contributions coming from the hyperon cuts and the two-pion cut, we investigate the K$\mathcal{N}$-scattering in $I=1$, s-state and in $I=0$, s-, $p_{\frac{1}{2}}$-, $p_{\frac{3}{2}}$-states. Qualitative agreement with the present experimental situation is obtained. Using crossing symmetry, the two-pion contribution in $\overline{\mathrm{K}}\mathcal{N}$-scattering is also considered.

1. – Introduction.

Experimentally, it is now known that the K$\mathcal{N}$-scattering in $I=1$ state is pure s-wave up to K-meson lab energy 315 MeV ([1,2]). On the other hand, the $I=0$ state scattering indicates p-wave interaction even at an energy 100 MeV ([1,2]). Besides, analysis of emulsion and counter data tend to indicate that the $I=0$ s-wave interaction is rather weak, while the $I=1$ s-wave interaction is strong and repulsive ([3]).

These features of K$\mathcal{N}$ low energy scattering have led the theoreticians to many speculations. In particular direct K-meson–pion interactions have been introduced by Barshay ([4]) and Yamaguchi ([5]). It has been pointed out by

([1]) M. F. Kaplon: *Proc. of the 1958 Ann. Int. Conf. on High Energy Physics at CERN*, p. 171; D. F. Davis, N. Kwak and M. F. Kaplon: *Phys. Rev.*, **117**, 846 (1960); T. F. Kycia, L. T. Kerth and R. G. Baender: *Phys. Rev.*, **118**, 553 (1960); O. R. Price, D. H. Stork and H. K. Ticho: *Phys. Rev.*, **119**, 1702 (1960).

([2]) Presented by H. K. Ticho: *Proc. of the 1960 Ann. Int. Conf. on High Energy Physics at Rochester*, p. 451.

([3]) R. H. Dalitz: *Proc. of the 1958 Ann. Int. Conf. on High Energy Physics at CERN*, p. 187.

([4]) S. Barshay: *Phys. Rev.*, **110**, 743 (1958).

([5]) Y. Yamaguchi: *Prog. Theor. Phys. Suppl.*, **11**, 37 (1959).

many authors that the behaviour of $K\mathcal{N}$-scattering is difficult to explain on the basis of scalar or pseudoscalar theories ([6,3]). Also application of fixed source dispersion theory, which has been successful in explaining low energy $\pi\mathcal{N}$-scattering ([7]), leads to predictions in contrast to experimental results ([8,9]).

Our purpose here is to assume an explanation of the isotropy of $K\mathcal{N}$-scattering in $I=1$ state, based on our present ideas of strong interactions and see how far we can then explain the other main features of $K\mathcal{N}$-scattering. Also, we shall investigate some consequences of our approach to the low energy $\bar{K}\mathcal{N}$-scattering.

2. – Basic assumption regarding $K\mathcal{N}$-scattering in $I=1$ state.

We begin with the consideration that for low energy $K\mathcal{N}$-scattering, the important left-hand contributions come from the hyperon cuts and the two-pion cut, which comes very close to the threshold. We approximate the two-pion cut by a single pole. This single-pole approximation may be regarded as a sharp two-pion resonance as considered by FRAZER and FULCO ([10]) and also by other authors to explain the nucleon's elecromagnetic structure and the small pion-nucleon phase shifts ([11,12]).

We denote the four-momenta of the incoming nucleon and K-meson by p_1 and q_1 while those of the outgoing particles by $-p_2$ and $-q_2$. The invariant Mandelstam variables are then

$$s = -(p_1+q_1)^2$$

$$\bar{s} = -(p_1+q_2)^2 \qquad \text{(we use } p\cdot q = \boldsymbol{p}\cdot\boldsymbol{q} - p_0 q_0\text{)}$$

$$t = -(p_1+p_2)^2 .$$

([6]) C. CEOLIN, V. DE SANTIS and L. TAFFARA: *Nuovo Cimento*, **12**, 502 (1959).

([7]) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957).

([8]) P. K. ROY: *Nucl. Phys.*, **20**, 417 (1960).

([9]) W. KROLIKOWSKI: *Proc. of the 1960 Ann. Int. Conf. on High Energy Physics at Rochester*, p. 477.

Capps has remarked that the contradiction that, with dispersion relations, the p-wave amplitudes in $I=1$ state come out larger than the corresponding amplitudes in $I=0$ can be removed, if one of the p-wave $\bar{K}\mathcal{N}$ absorptive amplitudes, occurring in the crossing term, gives sizable contribution: *Phys. Rev.*, **121**, 291 (1961).

([10]) W. R. FRAZER and J. R. FULCO: *Phys. Rev.*, **117**, 1609 (1960).

([11]) J. BOWCOCK, W. N. COTTINGHAM and D. LURIE: *Nuovo Cimento*, **16**, 918 (1960); *Phys. Rev. Lett.*, **5**, 386 (1960).

([12]) S. C. FRAUTSCHI: *Phys. Rev. Lett.*, **5**, 159 (1960).

The S-matrix element for $K\mathcal{N}$-scattering is given by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta(p_1 + q_1 + p_2 + q_2) \left(\frac{m_{\mathcal{N}}^2}{4E_1E_2\omega_1\omega_2}\right)^{\frac{1}{2}} \bar{u}(-p_2) T u(p_1) .$$

Writing the amplitude in the usual form (7)

$$T = -A + i\frac{\gamma(q_1 - q_2)}{2} B ,$$

let us denote the two-pion contribution to A and B by $A^{\pi\pi}(s, t)$, $B^{\pi\pi}(s, t)$.

Now, the single pole approximation of the two-pion interaction leads to the following forms ($I = 1$)

$$A^{\pi\pi}(s, t) = \left(s + \frac{t_r}{2} - \frac{\Sigma}{2}\right)\frac{a}{t - t_r} \qquad 1(a), \qquad \frac{\Sigma}{2} = m_{\mathcal{N}}^2 + \mu_{\mathrm{K}}^2 ,$$

$$B^{\pi\pi}(s, t) = \frac{b}{t - t_r} \qquad 1(b) ,$$

where we regard « a » and « b » as unknown parameters. The forms $1(a)$ and $1(b)$ are suggested from the works of Frautschi and Walecka (13) in $\pi\mathcal{N}$-scattering, of Lee (14) and of Ferrari, Frye, Pusterla (15) in $K\mathcal{N}$-, $\bar{K}\mathcal{N}$-scattering.

From $1(a)$ and $1(b)$ we can calculate the two-pion contributions in $I = 1$, $p_{\frac{1}{2}}$- and $p_{\frac{3}{2}}$-states to $K\mathcal{N}$-scattering. The corresponding amplitudes due to the hyperon cuts (the usual « Born approximations ») are also known. We now assume that the explanation of the isotropy of $K\mathcal{N}$-scattering in $I = 1$ state is that the two-pion amplitudes for $p_{\frac{1}{2}}$- and $p_{\frac{3}{2}}$-states cancel the corresponding Born amplitudes. This assumption at once gives us two equations for the parameters « a » and « b » in terms of the coupling constants $g_{\Sigma\mathrm{K}}^2$ and $g_{\Lambda\mathrm{K}}^2$ The equations are (in the static limit),

$$\text{(2i)} \quad a = \frac{1}{(2m_{\mathcal{N}}\mu_{\mathrm{K}} + t_r/2)} \frac{1}{2W_0} (g_{\Sigma\mathrm{K}}^2 + g_{\Lambda\mathrm{K}}^2) \frac{t_r}{\alpha(W_0)} \Big[(W_0 + m_{\mathcal{N}})(W_0 + m_{\Upsilon} - 2m_{\mathcal{N}}) \cdot \frac{t_r}{\alpha(W_0)} + (W_0 - m_{\mathcal{N}})(W_0 + 2m_{\mathcal{N}} - m_{\Upsilon})\Big] ,$$

$$\text{(2ii)} \quad b = \frac{1}{2W_0} (g_{\Sigma\mathrm{K}}^2 + g_{\Lambda\mathrm{K}}^2) \frac{t_r}{\alpha(W_0)} \Big[(W_0 + m_{\Upsilon} - 2m_{\mathcal{N}}) \frac{t_r}{\alpha(W_0)} - (W_0 + 2m_{\mathcal{N}} - m_{\Upsilon})\Big] ,$$

where $W_0 = m_{\mathcal{N}} + \mu_{\mathrm{K}}$, $\qquad \alpha(W_0) \equiv W_0^2 + m_{\Upsilon}^2 - 2(m_{\mathcal{N}}^2 + \mu_{\mathrm{K}}^2)$.

(13) S. C. Frautschi and J. D. Walecka: *Phys. Rev.*, **120**, 1486 (1960).
(14) B. W. Lee: *Ph. D. Thesis*, University of Pennsylvania, Oct. 1960 (unpublished).
(15) F. Ferrari, G. Frye and M. Pusterla: UCRL-9421.

3. – K$\mathcal{N}$-scattering in other states.

We now know the two parameters « a » and « b ». Therefore, we should be able to make definite predictions on s-wave scattering in $I=1$ state and on s-, $p_{\frac{1}{2}}$- and $p_{\frac{3}{2}}$-scattering in $I=0$ state. Denoting the two-pion contribution for s-wave in $I=1$ by $f_{0+}^{\pi\pi}(W)^{(I=1)}$, we find that this is repulsive. The Born term is also repulsive. Therefore, we expect repulsive interaction in this case. We defer the question of quantitative agreement with experimental results, in this case, for the moment and switch over to $I=0$.

Crossing symmetry shows that the two-pion $I=1$, $J=1$ resonant state gives $f_{0+}^{\pi\pi(0)}(W)=-3f_{0+}^{\pi\pi(1)}(W)$. The Born term in this case ($I=0$, $s_{\frac{1}{2}}$) is repulsive. Since $f_{0+}^{\pi\pi(1)}$ is repulsive, so $f_{0+}^{\pi\pi(0)}$ should be attractive and three times larger. This at once shows why we can expect rather weak s-wave $I=0$ interaction. To put this quantitatively, we find that the s-wave $I=0$ scattering length is given by

$$a_{00}=0.08\left[\left(\frac{3g_{\Sigma K}^2}{4\pi}-\frac{g_{\Lambda K}^2}{4\pi}\right)-\left(\frac{g_{\Sigma K}^2+g_{\Lambda K}^2}{4\pi}\right)3\,\frac{t_r}{\alpha(W_0)}\right]\text{fermi}\,, \tag{3}$$

where $\operatorname{tg}\delta_{00}=-a_{00}k$.

For $t_r=22\mu_\pi^2$ (Bowcock *et al.*) [11], $t_r/\alpha(W_0)=0.38$.

From (3) we then find that a suitably chosen ratio of $g_{\Sigma K}^2/g_{\Lambda K}^2$ will give agreement with results known from experiments [1,2,16,17].

Next, we proceed to calculate the $I=0$, $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ amplitudes, bearing in mind that as before the $I=0$ two-pion contribution is -3 times that of $I=1$ contribution. The two scattering lengths are given by

$$a_{03}^3=-0.04\cdot\frac{1}{11.5}\left[3\,\frac{g_{\Sigma K}^2}{4\pi}+\frac{g_{\Lambda K}^2}{4\pi}\right]\text{fermi}^3\,,\qquad(\operatorname{tg}\delta_{03}=-a_{03}^3k^3)\,, \tag{4}$$

$$a_{01}^3=\;\;0.02\cdot\frac{1}{11.5}\left[3\,\frac{g_{\Sigma K}^2}{4\pi}+\frac{g_{\Lambda K}^2}{4\pi}\right]\text{fermi}^3\,,\qquad(\operatorname{tg}\delta_{01}=-a_{01}^3k^3)\,. \tag{5}$$

Expressions (4) and (5) indicate that the $I=0$, $p_{\frac{3}{2}}$ is attractive and stronger than $I=0$, $p_{\frac{1}{2}}$ which is repulsive. These results are in qualitative agreement with the analysis of emulsion results [16] and quantitative agrement can be obtained by choosing suitable values of $g_{\Sigma K}^2$ and $g_{\Lambda K}^2$. However, our present knowledge of $g_{\Sigma K}^2$ and $g_{\Lambda K}^2$ as well as the status of emulsion data do not justify anything much beyond qualitative agreement.

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[16] M. A. Melkanoff, D. J. Prowse, D. H. Stork and H. K. Ticho: *Phys. Rev. Lett.*, **5**, 168 (1960).

We now come back to our discussion of $I=1$, s-wave scattering. In this case an effective range formula is usually applied to fit the experimental results ([17]). In our language, this means that we have not only the left-hand contributions, but also the right-hand contributions coming from the unitarity cut. The amplitude in this case will be given by ([18]),

$$k \operatorname{ctg} \delta_{10} = \frac{1 - \dfrac{W - W_1}{\pi} \displaystyle\int \left[f_{0+}^{\pi\pi(1)}(W') + f_{0+}^{B}(W') \right] \frac{k' \, \mathrm{d}W'}{(W' - W)(W' - W_1)}}{f_{0+}^{\pi\pi(1)}(W) + f_{0+}^{B(1)}(W)} , \tag{6}$$

where W_1 is the subtraction point and at this point the physical amplitude is put equal to $f_0^{\pi\pi(1)}(W) - f_0^{B(1)}(W)$. By adjusting the point W_1 and taking a suitable cut-off, we can now fit the experimental scattering length and effective range. For the purpose of illustration, taking $W_1 = W_0 = m_\Lambda + \mu_K$, we find the scattering length

$$a_{10} = 0.08 \left(1 + \frac{t_r}{\alpha(W_0)} \right) \left(\frac{g_{\Sigma K}^2 + g_{\Lambda K}^2}{4\pi} \right) \text{fermi} ,$$

and we get $r = 0.5$ fermi with a cut-off $k_{\max} = 3.6 \mu_\pi$. Here $k \operatorname{ctg} \delta_{10} = -1/a_{10} - \frac{1}{2} r k^2$ ($I=1$, s-wave).

4. – Two-pion contributions to $\bar{K}N$-interaction.

We now investigate the sign and magnitude of the two-pion contribution in $\bar{K}N$-scattering on the basis of our knowledge of this in KN-scattering. Use of crossing symmetry ([14]) shows that if we denote by $\bar{f}_l^{\pi\pi}(\overline{W})^{(\frac{1}{0})}$ the two pion contribution in $\bar{K} + N \to \bar{K} + N$ (the energy of this process being $\overline{W}$), then ([14])

$$\bar{f}_l^{\pi\pi}(\overline{W})^{(\frac{1}{0})} = f_l^{\pi\pi}(\overline{W})^{(\frac{1}{0})} ,$$

i.e. the two-pion pole gives the same contribution in $\bar{K}N$- and KN-scattering. This means in $I=1$ this contribution is repulsive and in $I=0$ it is three times larger and attractive. Ferrari, Frye and Pusterla ([19]) have tried to determine the sign of the two pion contribution following the procedure which has been applied in the πN case. They approximate the two-pion cut by

$$\operatorname{Im} G_{\pi\pi}(s) = R^1 \, \delta(s - a) , \tag{7}$$

$$G(s) = \frac{\sqrt{s}(E + m)}{k \operatorname{ctg} \delta - ik} = \sqrt{s}(E + m) f(W) ,$$

([17]) L. S. Rodberg and R. M. Thaler: *Phys. Rev. Lett.*, **4**, 372 (1960).

([18]) G. F. Chew and S. Mandelstam: *Phys. Rev.*, **119**, 467 (1960); G. Feldman, P. T. Matthews and A. Salam: *Nuovo Cimento*, **16**, 549 (1960).

([19]) F. Ferrari, G. Frye and M. Pusterla: *Phys. Rev. Lett.*, **4**, 615 (1960).

which gives the contribution

$$-\frac{1}{\pi}\frac{R^1}{s-a}\,, \tag{8}$$

to their amplitude $G(s)$. The position of their pole is given by $\sqrt{a}=9.6\mu_\pi$ which is quite near the physical threshold $W_0=10.2\mu_\pi$.

Let us now try to find out R^1 by equating (8) at threshold with that given by our $\bar{f}_0^{\pi\pi(1)}(\overline{W})$. We find

$$R^1 = -2.5\, f_{0+}^{\pi\pi(1)}(\overline{W}_0) M_{\mathcal{N}}^4\,, \tag{9}$$

and

$$f_0^{\pi\pi(1)}(\overline{W}_0) = -\left(\frac{g_{\Sigma\mathrm{K}}^2+g_{\Lambda\mathrm{K}}^2}{4\pi}\right)\frac{t_r}{\alpha(W_0)}\cdot 0.08\ \text{fermi}\,, \tag{10}$$

$$\left(\frac{t_r}{\alpha(W_0)}\begin{cases}=0.20 & \text{for } t_r=12\mu_\pi^2\\ =0.38 & \text{for } t_r=22\mu_\pi^2\end{cases}\right).$$

We see that R^1 has the positive sign as found by them. Taking $t_r=12\mu_\pi^2$, $\left((g_{\Sigma\mathrm{K}}^2+g_{\Lambda\mathrm{K}}^2)/4\pi\right)=10$, we get $R^1=0.40\, M_{\mathcal{N}}^4$ fermi. Since the coupling constants and the value of t_r are not well established, we cannot say much about the magnitude of R^1. However, it seems that our value of R^1 is possibly smaller by a factor of 2 than that of FERRARI *et al.* ([19]).

Finally, a word about the interpretation of the two-pion contribution in the physical $\bar{\mathrm{K}}\mathcal{N}$-scattering region is not out of place. FERRARI *et al.* ([15]) have defined the « interaction » as the discontinuity across the left-hand cut and then from eq. (7), considering the positive sign of R^1, they have interpreted the interaction as attractive in $I=1$ state. However, if we take the usual field theoretic definition ([20]), that an interaction is attractive (repulsive) if the phase shift due to it alone is positive (negative), then from (8) we find that their interpretation should be that the interaction is repulsive. This is essentially borne out by our eq. (10) for $f_0^{\pi\pi(1)}(\overline{W})$. For $I=0$, the interaction will be attractive and the net effect of two-pion interaction in $\mathrm{K}^-\mathrm{p}$-scattering is attractive. Another point worth making is that our amplitudes do not have the energy dependence $\propto\left(1/(W-\sqrt{a})\right)$ as indicated by (8), which is rather strong, because of the closeness of $\sqrt{a}$ to the physical threshold; so we should expect much less energy dependence of $k\,\mathrm{ctg}\,\delta$ than that of FERRARI *et al.* ([19,21]).

([20]) J. C. JACKSON and H. W. WYLD: *Phys. Rev. Lett.*, **2**, 355 (1959).

([21]) The phenomenological analysis of low energy $\bar{\mathrm{K}}\mathcal{N}$-scattering and reactions is based on the assumption that $k\,\mathrm{ctg}\,\delta$ can be regarded as a constant.

See R. H. DALITZ and S. F. TUAN: *Ann. Phys.*, **8**, 100 (1959); **10**, 307 (1960).

5. – Conclusion.

Summarizing, we can say that by assuming the isotropy of $K\mathcal{N}$-scattering in $I-1$ state as the result of the cancellation of the two-pion and the Born contributions in $p_{\frac{3}{2}}$- and in $p_{\frac{1}{2}}$-states, we have reproduced all the qualitative features of low energy $K\mathcal{N}$-scattering as indicated by our present experimental knowledge; also we have some insight into the two-pion contribution to $\bar{K}\mathcal{N}$-scattering. The solution of the experimental results, which agrees with all our theoretical considerations, is the D solution of the Rochester Conference [2] and the B solution of MELKANOFF *et al.* [16], characterized by weak $s_{\frac{1}{2}}$, attractive $p_{\frac{3}{2}}$ and repulsive $p_{\frac{1}{2}}$ in $I=0$ state and a repulsive pure s-wave interaction in $I=1$ state.

* * *

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RIASSUNTO (*)

Supponendo che l'isotropia dello scattering $K\mathcal{N}$ per $I=1$ sia ovuto alla compensazione dei contributi dell'onda p provenienti dai tagli dell'iperone e dal taglio dei due pioni, studiamo lo scattering $K\mathcal{N}$ nello stato $I=1$ e negli stati $I=0$, s, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$. Otteniamo una concordanza qualitativa con la situazione sperimentale attuale. Facendo uso della simmetria incrociata, consideriamo anche il contributo dei due pioni allo scattering $K\mathcal{N}$.

(*) *Traduzione a cura della Redazione.*

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Modified Static Equations for the Kπ-Interaction.

Effect of Pion-Pion Resonance.

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Summary. — The Cini-Fubini approximate version of the double dispersion representation is reduced to an integral equation in one variable for the Kπ-interaction. The resulting equation has the structure of a static theory equation but incorporates the effect of the $\pi+\pi \to K+\bar{K}$ channel on Kπ-scattering and satisfies the condition at the symmetry point precribed by the relativistic theory. A slight distortion of the crossing matrix renders possible an exact solution to the equation. The nature of the solution is discussed. Construction of the approximate unitary amplitudes for the Kπ-interaction which satisfy the crossing relations is discussed in the Appendix.

Introduction.

In a previous paper [1] by one of the present authors, a set of approximate solutions to the partial wave dispersion relations was derived for Kπ-scattering and for the process $\pi+\pi \to K+\bar{K}$. This set of solutions is characterized by the fact that the scattering phase shifts in both isotopic spin states of the Kπ system are identical. This implies the absence of the charge exchange scattering in Kπ-scattering and that the process $\pi+\pi \to K+\bar{K}$ proceeds only through the $T=0$ state. This situation resulted fron our assumption that the Kπ-interaction takes place through the renormalizable interaction Hamiltonian of the form [2]

$$H_{K\pi} = 4\pi\lambda(K^+, K)\pi^2 \tag{I.1}$$

[1] B. W. Lee: *Phys. Rev.*, **120**, 325 (1960): hereafter referred to as I.

[2] S. Barshay: *Phys. Rev.*, **109**, 2160 (1958); G. Costa and L. Tenaglia: *Nuovo Cimento*, **18**, 368 (1960).

and from the fact that the vanishing of the isotopic spin-flip amplitude, $A^{(1)}=0$, is compatible with the problem at hand [1]. The same result has also been obtained by GOURDIN, NOIROT and SALIN [3].

It was emphasized in the previous paper, however, that mechanisms other than that of eq. (I.1), such as the baryon-antibaryon intermediate states, would likely destroy the isotopic spin degeneracy of the $K\pi$ system. In such a case the virtual process $\pi+\pi\to K+\bar{K}$ below the physical threshold will conceivably be enhanced by the strong pion-pion interaction in the $T=1$ state [4,5]. Indeed, this assumption is the basis of the effective range analyses made for $K^-\mathcal{N}$-scattering by FERRARI, FRYE and PUSTERLA [6], and for $K^+\mathcal{N}$-scattering by LEE [7] and others [8].

In this paper we shall address ourselves to the investigations of the isotopic spin splitting in $K\pi$-scattering and of the effect of the strong pion-pion interaction in the $T=J=1$ state on $K\pi$-scattering. The point of view adopted in the present study should be made clear from the outset. From the viewpoint of perturbation theory the isotopic spin-flip amplitude would be zero unless one or more baryon loops inside a diagram are considered [1,9]. Our inability to deal with these and other higher mass intermediate states forces us to introduce into our theory another constant ξ, which will be defined later, in addition, to the « fundamental » coupling constant λ of eq. (I.1).

The substance of this paper consists of the reduction of the double dispersion representation for the $K\pi$-interaction to a single integral representation of the form discussed previously by CHEW and MANDELSTAM [10] and lately by CINI and FUBINI [11], and the solution of the resulting integral equation in the presence of a strong pion-pion interaction in the $J=T=1$ state in certain approximations. In the reduction of the double dispersion representation to a single integral representation, the crucial assumption is that in the low energy $K\pi$-scattering in the s-wave dominates and we need retain only the lowest angular momentum states under the dispersion integrals in the one-dimensional dispersion relation, which seems *a priori* reasonable as long as we are concerned with the low energy behavior. The partial wave decompo-

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(3) M. GOURDIN, Y. NOIROT and P. SALIN: *Nuovo Cimento*, **18**, 651 (1960).

(4) W. R. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959); *Phys. Rev.*, **117**, 1609 (1960).

(5) J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Phys. Rev. Lett.*, **5**, 386 (1960).

(6) F. FERRARI, G. FRYE and M. PUSTERLA: *Phys. Rev. Lett.*, **4**, 615 (1960).

(7) B. W. LEE: *Phys. Rev.* (in press); *Thesis* (unpublished).

(8) G. FRYE, R. L. WARNOCK, M. PUSTERLA and F. FERRARI: *Bull. Am. Phys. Soc. II*, **5**, 505 (1960).

(9) CHOU KUANG-CHAO: *Žurn. Èksp. Teor. Fiz.*, **38**, 1015 (1960).

(10) G. F. CHEW and S. MANDELSTAM: UCRL-9126 (unpublished).

(11) M. CINI and S. FUBINI: *Ann. Phys.*, **3**, 352 (1960).

sition of the resulting representation is carried out along the line of CGLN [12], and incorporating the suggestion of FRAZER [13] discussed for $\pi\mathcal{N}$-scattering. A change of variable, and an approximation similar to the static limit yield an integral equation in one variable which is analogous to the Chew-Low static equation [14] for the s-wave $\pi\mathcal{N}$-scattering save for the appearance of an extra term involving an integral over the absorptive parts of the $\pi+\pi \to K+\overline{K}$ amplitudes. A slight distortion of the crossing matrices renders possible an exact solution to this equation. Our method here is very similar to that of BOWCOCK, COTTINGHAM and LURIÉ [15] applied to $\pi\mathcal{N}$-scattering, but goes beyond theirs in that we are able to take into account the rescattering corrections (the effect of the right hand physical branch cuts).

We shall discuss the nature of the solution and, in particular, the possibility of an s-wave resonance in Kπ-scattering. In the Appendix we demonstrate the construction of the amplitudes for the Kπ-interaction which satisfy analyticity, and the (slightly distorted) crossing relations exactly and the unitarity condition in the lowest angular momentum states in all three channels. This is a generalization of the result of BLENKENBECLER [16] who constructed similar amplitude for the case where there is no splitting of the two isotopic spin states of the Kπ system.

1. – In the following presentation, we shall adhere to the notation of I as far as possible. The subtraction point will be altered, as will be shown, but the symmetry of the subtraction will not. That is, we require

$$\text{(1)} \qquad \begin{cases} s_0 = u_0 \,, \\ 2s_0 + t_0 = 2(M^2 + \mu^2) \,. \end{cases}$$

Let us concentrate on the isotopic spin-flip amplitude $A^{(-)}$. From the double dispersion representation follows the one dimensional dispersion relation with t fixed:

$$\text{(2)} \qquad A^{(-)}(s, u, t) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \mathrm{d}x \, \mathrm{Im}\, A^{(-)}(x, t) \left[\frac{1}{x - s} - \frac{1}{x - u} \right],$$

([12]) G. F. CHEW, M. L. GOLDBERGER, F. E. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957).

([13]) W. R. FRAZER: *Proceedings of the Tenth Annual Rochester Conference on High Energy Nuclear Physics* (to be published).

([14]) G. F. CHEW and F. E. LOW: *Phys. Rev.*, **101**, 1570 (1956).

([15]) J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Nuovo Cimento*, **16**, 918 (1960).

([16]) R. BLANKENBECLER: Princeton University preprint (1961).

where the antisymmetry of $A^{(-)}$ in s and u

$$A^{(-)}(s, u, t) = -A^{(-)}(u, s, t) \tag{3}$$

has been incorporated. In particular, this implies

$$A^{(-)}(s_0, s_0, t) = 0 . \tag{4}$$

We may project out the $J=1$ amplitude for reaction III from eq. (2) as suggested originally by MANDELSTAM [17],

$$A^{(-)} = \frac{3}{4}(s-u)B_1^{(-)}(t) + \\ + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \mathrm{d}x \, \mathrm{Im}\, A^{(-)}(x, t)\left[\frac{1}{x-s} - \frac{1}{x-u} - \frac{3}{4}\frac{s-u}{p^2q^2} Q_1\left(\frac{x+p_2+q^2}{2pq}\right)\right], \tag{5}$$

where Q_l is the Legendre function of the second kind. The partial wave amplitude $B_1^{(-)}$ satisfies the dispersion relation

$$B_1^{(-)}(t) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}t' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} . \tag{I.17}$$

and the discontinuity across the branch cut $0 \geqslant t > -\infty$ is given by eq. (I.18). It can be shown that if the lowest partial wave only is retained in $\mathrm{Im}\, A^{(-)}(x, t)$ in the integrand of eq. (5), *i.e.*,

$$\mathrm{Im}\, A^{(-)}(x, t) \approx \mathrm{Im}\, A_0^{(-)}(x) , \tag{6}$$

then the integral over the left cut in eq. (I.17) is cancelled by the last term in eq. (5) *provided* that the approximation of eq. (6) is also used in eq. (I.18) [10,13]. There results

$$A^{(-)}(s, u, t) \approx \frac{3}{4}(s-u)\frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'-t} + \\ + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \mathrm{d}x \, \mathrm{Im}\, A_0^{(-)}(x)\left[\frac{1}{x-s} - \frac{1}{x-u}\right]. \tag{7}$$

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[17] S. MANDELSTAM: *Phys. Rev.*, **115**, 1741 (1959).

We are here making the assumption that the low energy behavior of Kπ-scattering is dominated by the s-wave interaction. Consistency of this assumption can be checked *a posteriori* by evaluating the p-wave projection of the left-hand side of eq. (7) and showing it to be small.

A similar reduction can be applied to the amplitude $A^{(+)}$:

$$(8)\qquad A^{(+)}(s,u,t) \approx -\lambda + \frac{(t-t_0)}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}t' \frac{\operatorname{Im} B_0^{(+)}(t')}{(t'-t)(t'-t_0)} + \\ + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \mathrm{d}x \operatorname{Im} A_0^{(+)}(x)\left[\frac{s-s_0}{(x-s_0)(x-s)} + \frac{u-s_0}{(x-s_0)(x-u)}\right].$$

Writing

$$\frac{1}{x-s} - \frac{1}{x-u} = \frac{s-s_0}{(x-s_0)(x-s)} - \frac{u-s_0}{(x-s_0)(x-u)},$$

in eq. (7) and making suitable linear combinations of $A^{(+)}$ and $A^{(-)}$ according to eq. (I.7), (I-II). we obtain

$$(9)\qquad A^{(I)} \approx -\lambda + \frac{(t-t_0)}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}t' \frac{\operatorname{Im} B_0^{(+)}(t')}{(t'-t_0)(t'-t)} + \frac{3}{4}\beta_I \frac{s-u}{\pi}\int_{(2\mu)^2}^{\infty} \mathrm{d}t' \frac{\operatorname{Im} B_1^{(-)}(t')}{t'-t} + \\ + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\mathrm{d}x}{x-s_0}\left\{\operatorname{Im} A_0^{(I)}(x)\frac{s-s_0}{x-s} + \sum_{I'} \alpha_{II'} \operatorname{Im} A_0^{(I')}(x)\frac{u-s_0}{x-u}\right\},$$

where

$$(10)\qquad \alpha_{II'} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

$$(11)\qquad \beta_I = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

and they satisfy

$$(12)\qquad \begin{cases} \sum_{I''} \alpha_{II''}\alpha_{I''I'} = \delta_{II'}, \\ \sum_{I'} \alpha_{II'} = 1, \\ \sum_{I'} \alpha_{II'}\beta_{I'} = -\beta_I, \\ \det \alpha_{II'} = -1. \end{cases}$$

2. – Eq. (7) and (8) are the basis for deriving approximate crossing relations at the symmetry point [10]. As discussed by FRAZER for the pion-nucleon case [13], it is convenient to choose s_0 such that $t_0 = -2k_0^2$, *i.e.* $z_0 = \cos\theta_0 = 0$:

$$s_0 = \frac{1}{3}(M^2 + \mu^2 + 2\sqrt{M^4 - M^2\mu^2 + \mu^4}) . \tag{13}$$

If we assume, as seems reasonable in the low energy region, that the d- and higher waves are negligible,

$$\begin{cases} A(s, z) \approx A_0(s) + 3z A_1(s) , \\ A(s, z = 0) \approx A_0(s) . \end{cases} \tag{14}$$

Therefore one obtains from eq. (7) a first derivative condition

$$\frac{\partial}{\partial s} A_0^{(-)L}(s_0) = \frac{3s_0^2 + (M^2 - \mu^2)^2}{2s_0^2} \frac{3}{4} \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} B_1^{(-)}(t')}{t' + 2k_0^2} + \\ + \frac{s_0^2 + (M^2 - \mu^2)^2}{2s_0^2} \frac{\partial}{\partial s} A_0^{(-)R}(s_0) , \tag{15}$$

where the superscript $R(L)$ denoted the integral over the right (left) cut in eq. (I.10) and a similar expression for $A_0^{(-)}(s)$. One can indeed derive an infinite number of conditions at the symmetry point involving higher derivatives. When the left cut in eq. (I.10) is approximated by one or more poles, eq. (15) and similar equations will determine a consistent set of the locations and residues of the poles.

It is possible to set up derivative crossing relations at a new symmetry point, $t = t_0$, such that $s_0 = -(p_0^2 + q_0^2)$, for $B_1^{(-)}(t)$ and for $B_0^{(-)}(t)$ [the following discussion applies mainly to the former, however]. Though it may be possible to determine the normalization of $B_1^{(-)}$ [note that $B_1^{(-)}(t)$ satisfies the homogeneous integral equation. eq. (I.17)] through the derivative crossing relations for $B_0^{(-)}$ and eq. (15), so that $B_1^{(-)}(t)$ and $A_0^{(-)}(s)$ may be obtained consistently with eq. (7) in the low energy regions of reactions I and III, such a consistency requirement, neglecting the contributions from higher mass intermediate states, may be questioned on physical grounds. As we have emphasized previously, the splitting of the isotopic spin states in $K\pi$-scattering, and therefore the very existence of the non-vanishing isotopic spin-flip amplitude $A^{(-)}$ are due to the baryon-antibaryon and other higher mass intermediate states and the « size » of $B_1^{(-)}(t)$ is no doubt significantly influenced by the high energy behavior. In the following development, therefore, the normalization of $B_1^{(-)}(t)$ is treated as a phenomenological parameter that has to come from experiments.

The theorem of Fubini, Nambu and Wataghin [18] implies, in the present context, that the phase of $B_1^{(-)}(t)$ for $4\mu^2 \leqslant t \leqslant 16\mu^2$ is given by the phase shift of the pion-pion scattering in the $T = J = 1$ state. Therefore we may write

$$B_1^{(-)}(t) = F_\pi(t) \left\{ \frac{1}{\pi} \int_{16\mu^2}^{\infty} \mathrm{d}t' \, \frac{\mathrm{Im}\,[B_1^{(-)}(t') F_\pi^{-1}(t')]}{t' - t} + \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}t' \, \frac{\mathrm{Im}\, B_1^{(-)}(t')}{(t' - t) F_\pi(t')} \right\}, \tag{17}$$

where $F_\pi(t)$ is the pion form factor of Frazer and Fulco [4] normalized according to $F_\pi(0) = 1$. The lower limit of integration in the first term on the right of eq. (17) is lifted from $4\mu^2$ to $16\mu^2$. If the pion pair production is small in the pion-pion collision, this limit may be lifted further to a good approximation. If $F_\pi(t)$ represents a sharp resonance at $t = t_r$, we may further make an approximation in the spirit of the « steepest descent » method [19]:

$$B_1^{(-)}(t) \simeq F_\pi(t)\,[B_1^{(-)}(t_r)/F(t_r)] \equiv -\left(\frac{\xi}{4\pi}\right) F_\pi(t)\,, \tag{18}$$

where ξ is a phenomenological constant defined by

$$\xi = -4 \left\{ \int_{16\mu^2}^{\infty} \mathrm{d}t' \, \frac{\mathrm{Im}\,[B_1^{(-)}(t') F_\pi^{-1}(t')}{t' - t_r} + \int_{-\infty}^{0} \mathrm{d}t' \, \frac{\mathrm{Im}\, B_1^{(-)}(t')}{(t' - t_r) F_\pi(t')} \right\}. \tag{19}$$

The pole approximation, however, requires a considerable amount of labor, and the solution can be trusted only qualitatively. In the following sections, we shall describe an approximate reduction of eq. (9) to an integral equation of one variable, whose solution satisfies automatically the self consistency requirement, eq. (15), to a good approximation.

3. – The effect of the integral over $\mathrm{Im}\, B_0^{(+)}$ on $A^{(I)}$ in eq. (9) is small unless there is a strong pion-pion interaction in the $T = J = 0$ state. The integral over $\mathrm{Im}\, B_0^{(-)}$ contributes equally to both isotopic spin states and unless this is strongly energy-dependent, this term may be neglected, the main, energy-insensitive effect being incorporated in the subtraction constant λ. On the other hand, the integral over $\mathrm{Im}\, B_1^{(-)}(t)$ contributes differently in the $T = 0$ and $T = 1$ states and causes the splitting of the isotopic spin states, and this will be strongly enhanced if there is a pion-pion resonance in the $T = J = 1$ state.

(18) S. Fubini, Y. Nambu and V. Wataghin: *Phys. Rev.*, **111**, 329 (1958).
(19) This is the approximation used in ref. (7).

With the above assumption and by virtue of the second of eq. (14), we may write eq. (9) as

$$A_0^{(I)}(s) \approx -\lambda + \frac{3}{4}\beta_I \frac{s-u}{\pi} \int_{(2\mu)^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, B_1^{(-)}(t')}{t'+2k^2} - \tag{20}$$

$$+\frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{\mathrm{d}x}{x-s_0} \left\{ \mathrm{Im}\, A_0^{(I)}(x) \frac{s-s_0}{x-s} + \sum_{I'} \alpha_{II'}\, \mathrm{Im}\, A_0^{(I)}(x) \frac{u-s_0}{x-u(z=0)} \right\},$$

where

$$u(z=0) = 2(M^2+\mu^2) - s + 2k^2 . \tag{21}$$

Our aim here is to reduce eq. (20) to a tractable integral equation in one variable so that eq. (20) is approximated well in the low energy region $[(k/M)^2 \ll 1]$ of Kπ-scattering, rather than to use it in the pole approximation outlined in the previous section.

We choose our variable to be the incident pion energy in the laboratory system, ω:

$$s = M^2 + \mu^2 + 2M\omega . \tag{22}$$

We neglect the weak dependence of $u(z=0)$ on k^2, and choose $s_0 = M^2 + \mu^2$. In the case of $\xi = 0$ in eq. (18), there results

$$f_0^I(\omega) = -\lambda + \frac{\omega}{\pi} \int_{\mu}^{\infty} \frac{\mathrm{d}\omega'}{\omega'} \left\{ \frac{\mathrm{Im}\, f_0^I(\omega')}{\omega'-\omega} - \sum_{I'} \alpha_{II'} \frac{\mathrm{Im}\, f_0^{I'}(\omega')}{\omega'+\omega} \right\}, \tag{23}$$

where $f_0^I(\omega)$ is the amplitude $A_0^{(I)}$ expressed as a function of ω:

$$f_0^I(\omega) = 2 \frac{\omega+\hat{\omega}}{\sqrt{\omega^2-\mu^2}} \exp\left[i\delta_0^I(\omega)\right] \sin \delta_0^I(\omega) , \tag{24}$$

$$\hat{\omega} = (M^2+\mu^2)/2M .$$

Except for the inhomogeneous term and the kinematical factor in $f_0^I(\omega)$, eq. (23) has the structure identical to the Chew-Low equation for the static (charge symmetric) scalar pion theory. The amplitudes $f_0^I(\omega)$ satisfy the crossing relation of the Chew-Low type [14]:

$$f_0^I(\omega) = \sum_{I'} \alpha_{II'} f_0^{I'}(-\omega) . \tag{25}$$

We recognize that

$$f_0^{(\frac{1}{2})}(\omega) = f_0^{(\frac{3}{2})}(\omega) , \qquad \text{for } \xi = 0 , \tag{26}$$

is consistent with eq. (23). This gives

$$f_0^I(\omega) = -\lambda \left\{1 + \lambda \frac{\omega}{2\pi} \int_\mu^\infty \frac{d\omega'}{\omega'} \frac{\sqrt{\omega'^2 - \mu^2}}{\omega + \hat{\omega}} \left[\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega}\right]\right\}^{-1}, \tag{27}$$

$$I = \tfrac{1}{2} \text{ and } \tfrac{3}{2}.$$

This is essentially the solution we obtained in I, except that, here, the effect of the crossed channel II is taken into account correctly in the sense of the static theory. The effective range formula takes the form

$$\frac{\sqrt{\omega^2 - \mu^2}}{\omega + \hat{\omega}} \operatorname{ctg} \delta_0^{(I)} = -\frac{1}{\lambda} + \frac{1}{\lambda_c} + \frac{1}{2\pi} \left\{\sqrt{\omega^2 - \mu^2} \frac{2\omega}{\omega^2 - \hat{\omega}^2} \ln \frac{(\omega + \sqrt{\omega^2 - \mu^2}}{\mu} - \right. \tag{28}$$

$$\left. - (\omega - \mu)\left(\hat{\omega} - \frac{\mu^2}{M}\right)\left[\frac{1}{(\omega + \hat{\omega})(\hat{\omega} + \mu)} + \frac{1}{(\omega - \hat{\omega})(\hat{\omega} - \mu)}\right] \ln \frac{M}{\mu}\right\},$$

$$I = \tfrac{1}{2}, \tfrac{3}{2},$$

where λ_c is

$$\lambda_c = \frac{-2\pi}{\pi(\mu/\hat{\omega}) - (\hat{\omega} - \mu^2/M)(2\hat{\omega}/(\hat{\omega}^2 - \mu^2) - 2(\mu/\hat{\omega})) \ln (M/\mu)} \simeq -7.6. \tag{29}$$

For $\lambda \leqslant \lambda_c$, the solution, eq. (27), will develop a pole between $-\mu \leqslant \omega < \mu$, corresponding to a bound state fo the Kπ system. Therefore we require that

$$\lambda_c \leqslant \lambda. \tag{30}$$

It can be easily seen that the imaginary part of the denominator on the right of eq. (27) vanishes for imaginary ω. As ω increases in magnitude along the imaginary axis the denominator behaves asymptotically as

$$1 - \frac{\lambda}{\pi}\left\{\ln \frac{|\omega|}{\mu} + O(1)\right\}.$$

Hence if $\lambda > 0$, $f_0^{(I)}(\omega)$ will develop two poles at $\omega = \pm i|\omega_g|$ along the imaginary axis. Therefore we must require

$$-7.6 \leqslant \lambda \leqslant 0. \tag{31}$$

Eq. (31) implies that there is no resonance in Kπ-scattering as one can see from eq. (28). The qualitative nature of the solution, eq. (28), is not much different from that of eq. (I.13).

4. – In this section we wish to incorporate into eq. (23) the effect of the pion-pion resonance in the $T=J=1$ state. In doing this we shall insist upon the validity of the crossing relation, eq. (25) when the integral over $\mathrm{Im}\, B_1^{(-)}(t)$ is inserted in eq. (23). This restricts the possible approximations one can make in eq. (20). Consider the k^2 that appears in the integrand of the second term of eq. (20). We approximate ([20])

$$k^2 = \frac{M^2(\omega^2-\mu^2)}{2M\omega + M^2 + \mu^2} \approx \left(\frac{M}{M+\mu}\right)^2 (\omega^2-\mu^2)\,. \tag{32}$$

With the same approximation on $u(z=0)$ as before and eq. (31), eq. (20) may be reduced to

$$f_0^{(I)}(\omega) = -\lambda + \frac{3}{4}\frac{4M\omega}{\pi}\beta_I \int_{4\mu^2}^{\infty} \mathrm{d}t'\, \mathrm{Im}\, B_1^{(-)}(t') \left[t' + 2\left(\frac{M}{M-\mu}\right)^2 (\omega^2-\mu^2)\right]^{-1} + \tag{33}$$

$$+ \frac{\omega}{\pi}\int_{\mu}^{\infty} \frac{\mathrm{d}\omega'}{\omega'} \left\{ \frac{\mathrm{Im}\, f_0^{(I)}(\omega')}{\omega'-\omega} - \sum_{I'} \alpha_{II'}\, \frac{\mathrm{Im}\, f_0^{(I')}(\omega')}{\omega'+\omega} \right\}.$$

This equation satisfies the crossing relation, eq. (25), by virtue of eq. (12).

Eq. (33) has branch cuts from $\pm i\,[2\mu^2(M/(M+\mu))^2-\mu^2]^{\frac{1}{2}}$ to $\pm i\infty$ in addition to the ones from $\pm\mu$ to $\pm\infty$. While it is possible to solve eq. (33) approximately neglecting the left-hand cut on the real axis from $-\mu$ to $-\infty$, we have not been able to solve it exactly. If one distorts the crossing matrices $\alpha_{II'}$, β_I slightly in such a way that the relations, eq. (12) hold, then the exact solution to the crossing-distorted equation can be obtained.

We write

$$\alpha^0_{II'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta^0_I = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}. \tag{34}$$

The approximate crossing matrices $\alpha^0_{II'}$ and β^0_I clearly satisfy eq. (12), and their deviations from the original ones are small ([21]). In this approximation,

$$f^{(\frac{1}{2})}(\omega) = f^{(\frac{3}{2})}(-\omega)\,, \tag{35}$$

([20]) While the approximation $k^2=0$ preserves the crossing relation, eq. (25), the resulting equation is not compatible with the unitarity of $f_0^{(I)}(\omega)$.

([21]) Distortion of the crossing matrix in favor of an exact solution is not novel. A similar approximation has been used in the static $\pi\mathcal{N}$ theory by G. F. CHEW: *Encyclopedia of physics*, vol. 43 (Berlin, 1959).

and

$$(36)\qquad f_0^{(\frac{1}{2})}(\omega) \approx -\lambda + \frac{3}{4}\frac{4M\omega}{\pi}\cdot\frac{3}{2}\int_{4\mu^2}^{\infty} \mathrm{d}t' B_1^{(-)}(t')\left[t' + 2\left(\frac{M}{M+\mu}\right)^2(\omega^2-\mu^2)\right]^{-1} + \\ + \frac{\omega}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{d}\omega'}{\omega'}\left\{\frac{\operatorname{Im} f_0^{(\frac{3}{2})}(\omega')}{\omega'-\omega} - \frac{\operatorname{Im} f_0^{(\frac{3}{2})}(\omega')}{\omega'+\omega}\right\}.$$

The structure of eq. (36) is, except for the branch cuts along the imaginary axis, identical with the equation for the Lee-Serber model ([22]) and the exact solution to eq. (36) can be obtained. The customary N/D method reduces eq. (36) to a linear integral equation for either N or D, which can be solved by a number of standard methods available, once $\operatorname{Im} B_1^{(-)}(t)$ is known.

To effect an easily manageable solution of eq. (36), we make the approximation of a sharp pion-pion resonance. According to Bowcock, Cottingham and Lurié ([15]), we may write

$$\operatorname{Im} F_\pi(t) \approx \pi\,\delta(t-t_r)$$

in the limit of a zero-width resonance, where t_r is the location of the resonance. In this approximation

$$(37)\qquad \operatorname{Im} B_1^{(-)}(t) \simeq -\frac{\xi}{4\pi}\pi\,\delta(t-t_r)\,,$$

and the branch cuts along the imaginary axis are replaced by two poles located at $\pm i\left[(t_r/2)\big(M/(M+\mu)\big)^2-\mu^2\right]^{\frac{1}{2}} = \pm iz$.

Writing $f_0^{(\frac{1}{2})}(\omega)$ as $N(\omega)/D(\omega)$, we assign the singularities N and D as in I, and we obtain

$$(38)\qquad \begin{cases} N(\omega) = -\lambda - \dfrac{9}{16\pi}\,\xi\,\dfrac{(m+\mu)^2}{m}\,\dfrac{\omega}{2iz}\left[\dfrac{D(iz)}{\omega-iz} - \dfrac{D(-iz)}{\omega+iz}\right], \\ D(\omega) = 1 - \dfrac{\omega}{2\pi}\displaystyle\int_{\mu}^{\infty}\frac{\mathrm{d}\omega'}{\omega'}\,\frac{\sqrt{\omega'^2-\mu^2}}{\omega'+\hat{\omega}}\,N(\omega')\left(\frac{1}{\omega'-\omega} - \frac{1}{\omega'+\omega}\right), \\ \dfrac{\sqrt{\omega^2-\mu^2}}{\omega+\hat{\omega}}\operatorname{ctg}\delta_0^{(\frac{1}{2})} = N^{-1}(\omega)\operatorname{Re} D(\omega)\,. \end{cases}$$

Note that while the imaginary part of $D(\omega)$ of eq. (38) vanishes along the imaginary axis, this is a consequence of eq. (34), and the imaginary part of $D(\pm i|\omega|)$ need not be zero when the correct crossing matrix, eq. (10), is used.

([22]) L. Castillejo, R. H. Dalitz and F. J. Dyson: *Phys. Rev.*, **101**, 453 (1956).

The quantity

$$\frac{\sqrt{\omega^2-\mu^2}}{\omega+\hat{\omega}}\ \mathrm{ctg}\ \delta_0^{(\frac{1}{2})}\ ,$$

is plotted against ω in Fig. 1 and 2 for some sets of λ, ξ and t_r ([23]). We note that

(39) $$f_0^{(\frac{1}{2})}(\omega, -\xi) = f_0^{(\frac{3}{2})}(\omega, \xi)$$

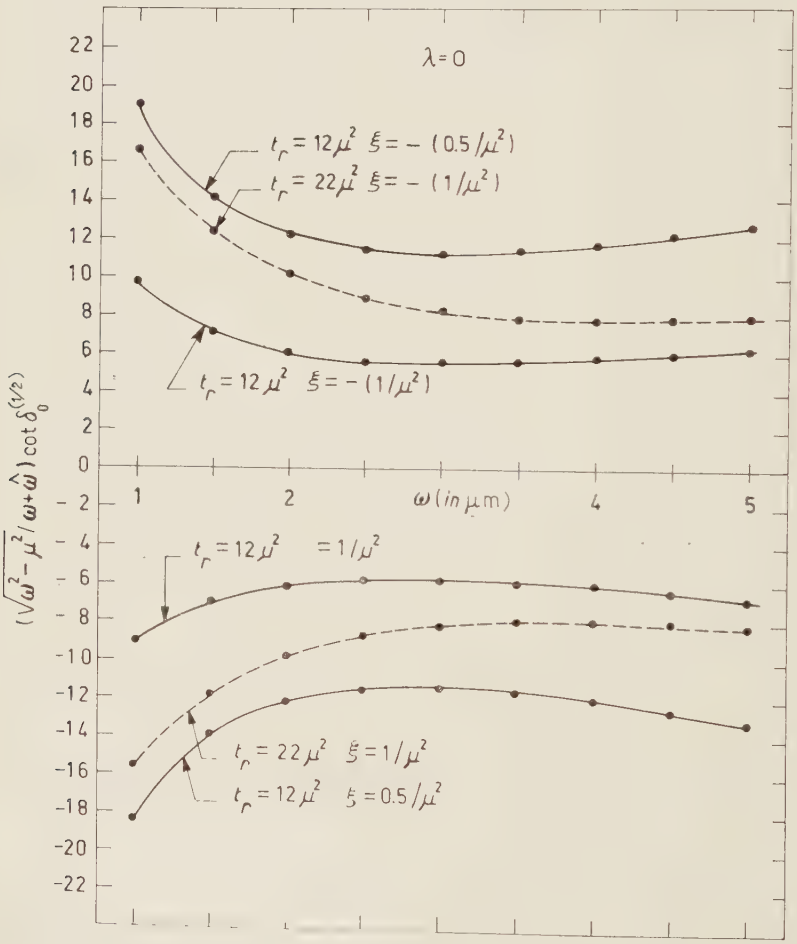

Fig. 1. – $\left(\sqrt{\omega^2-\mu^2}/(\omega+\hat{\omega})\right)$ ctg $\delta_0^{(\frac{1}{2})}$ *vs.* ω, for $\lambda=0$, the dot lines correspond to the choice $t_r=22\mu^2$, the black lines to $t_r=12\mu^2$.

([23]) $t_r\approx 12\mu^2$ is the value of ref. ([4]): $t_r\approx 22\mu^2$ is the value deduced in references ([5]) and ([15])

in our approximation, and therefore we have shown the plot only for the $T=\frac{1}{2}$ state, for ξ's of equal magnitude and opposite signs. We have taken $\lambda=0$ (Fig. 1) and -4 (Fig. 2). The energy dependence of the s-wave phase

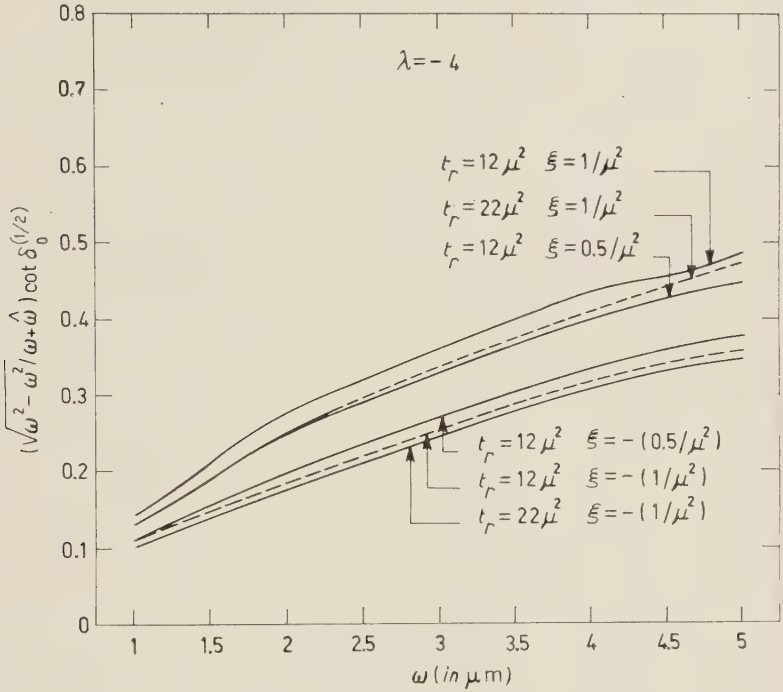

Fig. 2. – $(\sqrt{\omega^2-\mu^2}/(\omega+\hat{\omega}))$ ctg $\delta_0^{(\frac{1}{2})}$ *vs.* ω, for $\lambda=-4$, the dot lines correspond to $t_r=22\mu^2$, the black lines to $t_r=12\mu^2$.

shift in K^+p scattering indicates, for $t_r=12\mu^2$, $\xi=0.5$ to $1.0\,\mu^2$ (7). For $\lambda=0$, the splitting of the two isotopic spin states are pronounced, $\delta_0^{(\frac{1}{2})}$ and $\delta_0^{(\frac{3}{2})}$ being of about the same magnitude, but of opposite signs. For values of ξ compatible with the data on K^+p scattering, neither state has a low-energy ($\omega \lesssim 4\mu$) resonance in the present theory. For $\lambda=-4$, the effect of the « contact potential » of strength λ overshadows the effect of the two-pion (in the $T=J=1$ state) exchange, and the isotopic spin splitting is small for values of ξ quoted here.

5. – Concluding remarks.

In the present paper, the one-dimensional approximate version of the double dispersion representation for the Kπ-interaction is further reduced to an integral equation in one variable, which has a strong resemblance to the

static Chew-Low equation. Let us reiterate the philosophy behind this reduction. In the applications of the Mandelstam representation so far made, the approximate single integral representation has been used, explicitly or impliedly ([11]), except in the proposed program of the strip approximation ([24]). This approximate representation has of course only a limited range of validity and is therefore reliable only for those values of the kinematical variables for which the singularities associated with higher mass intermediate states are distant. For these values of the variables, we can make a further approximation on the kinematical variables which reduces the Cini-Fubini version ([11]) of the Mandelstam representation to an integral equation in one variable. Although the resulting integral equation is very similar to the static equation, the approximation employed is not $(\mu/m) \to 0$. Furthermore, our integral equation, eq. (33), differs from the static equation in the following two respects: 1) the effect of the low energy behavior of the process $\pi + \pi \to K + \bar{K}$ on $K\pi$-scattering is included; 2) the subtraction demanded by the relativistic dispersion theory is correctly incorporated, *i.e.* the condition $A^{(-)}(s, u) = A^{(-)}(u, s)$ implies $f_0^{(\frac{1}{2})}(0) \simeq f_0^{(\frac{3}{2})}(0)$. Our method is similar to those of CGLN ([12]) and Bowcock, Cottingham and Lurié ([15]), but contrary to assuming the dominant contribution to the absorptive part from the narrow (3.3)-resonance in the pion-nucleon case, we have solved the integral equation, thereby estimating the rescattering corrections correctly.

For values of ξ deduced from K^+p-scattering in a previous work, we find that the present model of $K\pi$-scattering does not predict a low-energy resonance in either isotopic spin state. Implicit in our reduction of the quasi-static equation is the assumption that in the low energy $K\pi$-scattering only the s-wave is important. For the solutions shown in Fig. 1 and 2, the contribution from the crossed cut to the p-wave amplitude is very small, as is the contribution from the $\pi\pi$ cut, so that the consistency of the s-wave dominance in the present model is verified.

If there exists an unstable scalar particle, K′, of mass greater than $M + \mu$, of opposite parity relative to K and of strangeness $+1$, as Bernstein ([25]) *et al.* predict, then this will manifest itself as a resonance in $K\pi$ s-state scattering. Unpublished data of M. L. Good *et al.* ([26]) seem to indicate such a resonance in the $T = \frac{1}{2}$ state. If the mechanism of the resonance is indeed that envisaged by Bernstein *et al.*, we must add a Castillejo-Dalitz-Dyson

([24]) G. F. Chew and S. Frautchi: *Phys. Rev. Lett.*, **5**, 580 (1961).

([25]) J. Bernstein and S. Weinberg: *Phys. Rev. Lett.*, **5**, 481 (1960); also references quoted therein.

([26]) M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho and S. G. Wojcicki: *Phys. Rev. Lett.*, **5**, 11, 520 (1960), and private communications.

pole ([22]) to the right-hand side of the equation for $D(\omega)$ in eq. (38), which will surely produce a resonance in Kπ-scattering.

* * *

We have enjoyed discussions with Professors R. BLANKENBECLER, A. KLEIN and W. SELOVE on the topics dealt with in this paper.

APPENDIX

BLANKENBECLER ([16]) has constructed an appropriate amplitude for the Kπ-interaction ($\pi+\mathrm{K}\to\pi+\mathrm{K}$, $\pi+\pi\to\mathrm{K}+\overline{\mathrm{K}}$) treating the crossing relations exactly here taking into account only the lowest possible partial waves in lack of the three channels for the case where there is no splitting of the two isotopic spin states of the Kπ system. For ease in reading we shall transcribe his result:

$$\text{(A.1)}\quad A^{(+)}(s,t,u) = -\lambda \exp\left[\Delta^{(+)}(t)\right]\left\{1+\lambda\frac{1}{\pi}\int\limits_{(m+\mu)^2}^{\infty}\frac{\mathrm{d}x\,\varrho(x)f^{(+)}(x)}{x-s_0}\left[\frac{s-s_0}{x-s}+\frac{u-s_0}{x-u}\right]\right\}^{-1},$$

where

$$\text{(A.2)}\qquad \Delta^{(+)}(t) = \frac{(t-t_0)}{\pi}\int\limits_{2(\mu)^2}^{\infty}\frac{\delta_0^{\pi\pi}(t')}{(t'-t_0)(t'-t)},$$

$\delta_0^{\pi\pi}$ being the $\pi\pi$ phase shift in the $T=J=0$ state, and

$$\text{(A.3)}\qquad \begin{cases} \varrho(s) = \dfrac{k}{\sqrt{s}}, \\ f^{(+)}(s) = \dfrac{1}{2}\displaystyle\int_{-1}^{1}\mathrm{d}z\,\exp\left\{\Delta^{(+)}[t(s,z)]\right\} \approx \exp\left[\Delta^{(+)}[t(s,0)]\right]. \end{cases}$$

If we set $\delta_0^{\pi\pi}=0$ and project out $A_0^{(+)}(s)$ approximately according to eq. (14), making at the same time the approximation

$$U(z=0) \approx m^2+\mu^2-2m\omega,$$

eq. (27) follows from eq. (A.1).

A similar construction is possible in the case of nonvanishing $A^{(-)}$, if the approximate crossing matrices $\alpha_{II'}^0$, β_I^0 defined in eq. (34) are used. Instead of a derivation, we shall present the result first, and justify it subsequently.

$$\text{(A.4)}\quad A^{(I)}(s,t,u) = -\lambda\exp\left[\Delta^{(+)}(t)\right]\left[D^{(+)}(s,u)\right]^{-1} -$$

$$-\beta_I^0(s-u)\frac{3}{16\pi}\frac{\xi F_\pi(t)}{\left[D^{(+)}(s,u)\right]^2}\left\{1-\frac{\beta_I^0}{\pi}\int\limits_{(m+\mu)^2}^{\infty}\mathrm{d}x\,\frac{\varrho(x)f^{(-)}(x)}{x-s_0}\left[\frac{s-s_0}{x-s}-\frac{u-s_0}{x-u}\right]\right\}^{-1},$$

where

$$D^{(+)}(s,u) = 1 + \frac{\lambda}{\pi}\int\limits_{(m+\mu)^2}^{\infty} \mathrm{d}x\, \frac{\varrho(x) f^{(+)}(x)}{x-s_0}\left[\frac{s-s_0}{x-s} + \frac{u-s_0}{x-u}\right], \tag{A.5}$$

and

$$f^{(-)}(s) = \frac{1}{2}\int\limits_{1}^{1} \mathrm{d}z \left(-\frac{3\xi}{16\pi}\right) \frac{F_\pi[t(s,z)]}{|D^{(+)}[s,u(s,z)]|^2}\,[s-u(s,z)] \approx \tag{A.6}$$

$$\approx -\frac{3\xi}{16\pi}\,\frac{F_\pi[t(s,0)]}{|D^{(+)}[s,u(s,0)]|^2}\,[s-u(s,0)]\,.$$

It is clear that $A^{(I)}(s,t,u)$ of eq. (A.4) possesses the correct analytic properties prescribed in eq. (9). These amplitudes satisfy

$$A^{(\frac{1}{2})}(s,t,u) = A^{(\frac{3}{2})}(u,t,s)\,, \tag{A.7}$$

which is the analogue of eq. (35). Again we project out according to eq. (14). The result has the form

$$A^{(I)}(s) = \frac{\exp[i\alpha]\sin\alpha}{\varrho(s)} + \exp[2i\alpha]\,\frac{\exp[i\beta^{(I)}]\sin\beta^{(I)}}{\varrho(s)} = \tag{A.8}$$

$$= \varrho(s)^{-1}\exp[i(\alpha+\beta^{(I)})]\sin(\alpha+\beta^{(I)})\,, \qquad s \geqslant (m+\mu)^2\,,$$

and therefore $A^{(I)}(s)$ satisfies the unitary condition in the physical region of channel I. Indeed the form of eq. (A.8) is a general feature of the elastic partial wave amplitude when there are two scattering «mechanisms» operative ([27]). This separation of the T-matrix element is familiar in the Coulomb-nuclear interference ([27]), and in the resonance scattering where the «potential» scattering part and the resonance scattering part (Breit-Wigner formula) interfere in the form of eq. (A.8) ([28]).

Let us compute $A^{(I)}(s,t,u)$ from $A^{(I)}(s,t,u)$ of eq. (A.4) according to the modified crossing relation of eq. (35). First

$$A^{(-)} = \tfrac{1}{3}[A^{(\frac{1}{2})} - A^{(\frac{3}{2})}] = -(s-u)\,\frac{3}{16\pi}\,\frac{\xi F_\pi(t)}{[D^{(+)}(s,u)]^2}\cdot \tag{A.9}$$

$$\cdot\left\{1 - \frac{9}{17}\left|\int \mathrm{d}x\,\frac{\varrho(x) f^{(-)}(x)}{x-s_0}\left(\frac{s-s_0}{x-s} - \frac{u-s_0}{x-u}\right)\right|^2\right\}^{-1} \approx 3\cos\varphi\cdot pq\left[-\frac{\xi}{4\pi}F_\pi(t)\right].$$

([27]) S. Drell and F. Zachariasen: *Phys. Rev.*, **105**, 1407 (1957).

([28]) See, for instance, H. Feshbach: *Ann. Phys.*, **5**, 357 (1958); B. W. Lee and A. Klein: *Nuovo Cimento*, **13**, 891 (1959).

The expression, eq. (A.9) is antisymmetric in s and u and has the same phase for $F_\pi(t)$. The last approximate identity follows from expanding $A^{(-)}(s, t, u)$ in powers of $\cos\varphi$ and agrees with eq. (18) and (I 8III). Next consider $A^{(+)}(s, t, u)$:

$$(A.10)\qquad A^{(+)} = \tfrac{1}{2}[A^{(\frac{1}{2})} + A^{(\frac{3}{2})}] = -\lambda \exp[\Delta^{(+)}(t)][D^{(+)}(s, u)]^{-1} -$$

$$-\frac{9}{4}(s-u)\,\frac{3}{16\pi}\,\frac{\xi F_\pi(t)}{[D^{(+)}(s, u)]^2}\,\frac{\dfrac{1}{\pi}\displaystyle\int_\infty \mathrm{d}x\,\frac{\varrho(x) f^{(-)}(x)}{x - s_0}\left(\frac{s - s_0}{x - s} - \frac{u - s_0}{x - u}\right)}{1 - \dfrac{9}{4\pi^2}\left|\displaystyle\int_\infty \mathrm{d}x\,\frac{\varrho(x) f^{(-)}(x)}{x - s_0}\left(\frac{s - s_0}{x - s} - \frac{u - s_0}{x - u}\right)\right|^2}.$$

This is symmetric in s and u and is normalized to $-\lambda$ at $s = u = s_0$, $t = t_0$. Also

$$(A.11)\qquad B_0^{(+)}(t) \approx A^{(+)}[s(t, \cos\varphi = 0), t] = -\lambda \exp[\Delta^{(+)}(t)]\,,$$

which is exactly eq. (I 19).

RIASSUNTO (*)

La versione approssimata di Cini-Fubini della rappresentazione della doppia dispersione viene ridotta ad una equazione integrale in una variabile per la interazione K-π. L'equazione, che risulta, ha la struttura di una equazione della teoria statica ma incorpora l'effetto del canale $\pi+\pi \to K+\overline{K}$ sullo scattering K-π e nel punto di simmetria soddisfa la condizione prescritta dalla teoria relativistica. Una piccola distorsione della matrice d'incrocio rende possibile la soluzione esatta dell'equazione. Si discute la natura della soluzione. Nell'Appendice si discute la costruzione delle ampiezze unitarie approssimate per l'interazione K-π che soddisfano alla relazione di incrocio.

(*) *Traduzione a cura della Redazione.*

Charge and Current Densities in the Faraday Disc.

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Summary. — The current distribution in a Faraday disc (homopolar generator) is investigated by solving approximately the magnetohydrodynamical equations. The current density streamlines turn out to be spirals, with the possibility of a change in sense in the case that the magnetic field and the rotation axis are parallel. A striking feature is the absence of solutions showing electrical neutrality. The induced charge density is proportional to $B\omega$ (B=applied magnetic field, ω=angular velocity of the disc). Then, from a more fundamental approach, the Boltzmann equation for the Faraday disc is solved. This extends Sommerfeld's work on metals in a magnetic field to the case of moving metals.

1. – Introduction.

When a copper disc is spun transverse to a magnetic field, an electric current is induced in the disc which flows in some manner from its center to its rim, or vice-versa, depending on the orientation of the rotation axis relative to the magnetic field. If a complete circuit is formed by attaching one end of a wire to the center of the disc and the other end to a sliding contact on its rim, a steady current flows; and one has the *Faraday disc*, or homopolar generator in its simplest form. The purpose of this paper is to examine theoretically the detailed form of the charge and current densities in the disc.

One notices first that the simple ideas governing currents induced in wires (and other essentially one dimensional conductors) moving in magnetic fields are not adequate for this three (or at least two) dimensional problem. For instance, in the former theories it was sufficient to consider the « dynamo current » density

$$\boldsymbol{j} = \varkappa(\boldsymbol{v} \times \boldsymbol{B}) \tag{1.1}$$

driven by the motional emf where $\boldsymbol{v}$ is the velocity of the element of the conductor at the point in question, $\varkappa$ is its conductivity, and $\boldsymbol{B}$ is the applied magnetic field. But if one attempts to apply (1.1) to the Faraday disc, a radial current proportional to r is found, and an immediate contradiction of charge conservation ensues.

One is led to go back to the more fundamental and general Ohm's Laws derived in the theory of fully ionized gases ([1]), for instance Sp (2-12). But even this is too specialized for our present purpose, because, among other things it assumes electrical neutrality, hence possible convective currents are left out of account. In Section 2 we find that there are no solutions to this problem with electrical neutrality. A remark which will be useful later, in Section 3, is the following: current densities derived under the assumption of electrical neutrality like Sp (2-12) can be interpreted as the *conductive part* in a general, non electrically neutral, context provided that the ion motion is identified with the center-of-mass motion (*i.e.*, that the electron contribution to the mass flow is neglected). Another reason for bypassing the generalized Ohm's Law is found in the special features of our problem. Here the ion « gas » is tightly bound in a metallic lattice and therefore the average ion motion may be taken as prescribed. Thus it is natural to study the electron motion alone, and only at the end to form the current density.

Our method in Section 2 is to treat the coupled macroscopic equations of transfer for the electron gas and Maxwell equations for the accompanying electromagnetic field. We treat the steady state and take the temperature to be uniform. The main approximations are the following.

a) The usual hydrodynamic neglect of non-linear terms $= O(v^2)$ in the electron mean velocity $\boldsymbol{v}$; the neglect of the effect of the induced magnetic field $\boldsymbol{B}_1$ relative to the effect of the applied field $\boldsymbol{B}$ on the electron motion. These two approximations must of course be *a posteriori* justified once the solutions are obtained.

b) The mean ion motion $\boldsymbol{v}_i$ is prescribed as

$$\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r} \,, \tag{1.2}$$

where $\boldsymbol{\omega}$ is the constant angular velocity of the disc.

c) The collision term in the momentum transfer equations will be written

$$\boldsymbol{P} = - nmv(\boldsymbol{v} - \boldsymbol{\omega} \times \boldsymbol{r}) \,, \tag{1.3}$$

where $\boldsymbol{P}$ = total momentum transferred to the electrons per unit volume per

([1]) See, for example, L. SPITZER: *Physics of Fully Ionized Gases* (New York, 1956); Sect. 2.2. This work will be referred to hereafter by the prefix Sp.

unit time by collisions with ions; $n =$ electron number density; $m =$ electron mass; $\boldsymbol{v} =$ average electron velocity; $\nu =$ some average collision frequency of an electron with ions. This corresponds to using the « naive » theory for the electrical resistivity η,

$$\eta = \frac{m\nu}{nq^2}\,, \tag{1.4}$$

($q(>0) =$ emu electron charge $\approx 1.6\cdot 10^{-20}$) in which it is inversely proportional to the first power of n and directly proportional to some suitable average ν over electron velocities c of the collision frequency $\nu(c)$. A more refined expression for the resistivity could easily be used without making any essential changes in the theory presented below.

In Section **3** the elementary theory given in Section **2** is supplemented by a more fundamental treatment. The Boltzmann equation for the velocity distribution function of the electron gas is solved by the method of Chapman and Cowling ([2]). The fundamental treatment adds nothing qualitatively new to the results of the elementary theory except for a single important item: the correct value $c = w$ to substitute into the strongly velocity dependent collision frequency $\nu(c)$ to get the « suitable average » ν is indicated. As is well known, owing to the advanced degeneracy of the electron gas, this « average velocity » w is much greater than the average thermal velocity the electrons would have at this temperature in the absence of degeneracy. This work is an extension of Sommerfeld's treatment ([3]) of electron diffusion in a metal *at rest* in the presence of a magnetic field. Evidently for the Faraday disc the centroid motion is the essential feature.

2. – The elementary theory.

We align the positive z axis along $\boldsymbol{\omega}$, the constant angular velocity of our thin metal disc of radius R and thickness $\varepsilon \ll R$. The constant uniform applied magnetic field $\boldsymbol{B} = B\delta\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector along Oz and $\boldsymbol{B}$ is parallel or antiparallel to $\boldsymbol{\omega}$ according as $\delta = +1$ or -1. With the approximations and notation of Section **1** the macroscopic equations for the electron gas and the electromagnetic field are

$$\begin{aligned} nq(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + \nabla p_e + nm\nu(\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{r}) &= 0\,, \\ \nabla\cdot(n\boldsymbol{v}) &= 0\,, \\ \nabla\cdot\boldsymbol{E} &= 4\pi c^2 q(n_i Z - n)\,, \\ \nabla\times\boldsymbol{E} &= 0\,. \end{aligned} \tag{2.1}$$

([2]) S. Chapman and T. Cowling: *The Mathematical Theory of Non-Uniform Gases* (Cambridge, 1952). This work will be referred to by the prefix CC.

([3]) See CC, Sect. **18**.5.

We use emu units. $n_i = \text{const}$ is the number density of the ions of charge Zq. The electron pressure p_e is some known function of n. A metal is a dense and quite degenerate (at N. T. P.) Fermi gas, so we take

$$p_e \approx \frac{1}{5}\left(\frac{3}{8\pi}\right)^{\frac{2}{3}} \frac{h^2}{m} n^{\frac{5}{3}} \left\{1 + \frac{5\pi^2}{12}(2mkT)^2 \left(\frac{8\pi}{3nh^3}\right)^{\frac{4}{3}}\right\}. \tag{2.2}$$

We shall consider only the case of uniform temperature.

By cylindrical symmetry we have $E_\theta = 0$ and all quantities are independent of θ, where r, θ, z are cylindrical co-ordinates with origin at the center of the disc. Therefore the eq. (2.1) become

$$\left\{\begin{array}{ll} a) & E_r + v_\theta B\delta + \dfrac{1}{q}\dfrac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)' + \dfrac{m\nu}{q} v_r = 0\,, \\ b) & -v_r B\delta + \dfrac{m\nu}{q} v_\theta - \dfrac{m\nu}{q}\omega r = 0\,, \\ c) & E_z + \dfrac{1}{q}\dfrac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)^{\cdot} + \dfrac{m\nu}{q} v_z = 0\,. \\ d) & \dfrac{1}{r}(rnv_r)' + (nv_z)^{\cdot} = 0\,, \\ e) & \dfrac{1}{r}(rE_r)' + \dot{E}_z = 4\pi e^2 q(n_i Z - n)\,, \\ f) & \dot{E}_r - E_z' = 0\,. \end{array}\right. \tag{2.3}$$

Here $(\,)'$ and $(\,)^{\cdot}$ denote $\partial/\partial r(\,)$ and $\partial/\partial z(\,)$ respectively. These are six equations for the six unknowns v_r, v_θ, v_z, n, E_r, and E_z. The method will be to eliminate unknowns between equations, finally getting a single differential equation for one of them. The most convenient one proves to be n. Solve for v_θ from b and substitute this expression into a. Then multiply a by r, differentiate this equation with respect to r, and add to the z derivative of c. $\nabla \cdot \boldsymbol{E}$ can be eliminated by e, and one obtains the equation

$$4\pi e^2 q(n_i Z - n) + \frac{1}{qr}\left[r\frac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)'\right]' + \frac{1}{q}\left[\frac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)^{\cdot}\right]^{\cdot} + \tag{2.4}$$

$$+ \frac{m\nu}{q}\Omega^{-1}\frac{1}{r}(rv_r)' + \frac{m\nu}{q}\dot{v}_z + 2\delta B\omega = 0\,,$$

where

$$\Omega^{-1} = 1 + \left(\frac{\omega_c}{\nu}\right)^2, \qquad \omega_c = \frac{qB}{m}\,.$$

Similarly f with the help of a and b gives

$$\dot{v}_r \, \Omega^{-1} = v_z' \tag{2.5}$$

or, the velocity field is irrotational in the limit $\Omega \to 1$ of weak magnetic fields. The disc is very thin, and we try for a solution with laminar flow

$$v_z = 0 \, . \tag{2.6}$$

Then d can be immediately integrated to yield

$$v_r = \frac{f(z)}{rn} \, , \tag{2.7}$$

where $f(z)$ is an arbitrary function of z. By (2.5) v_r is no function of z in the case of laminar flow. Thus (2.7) states than n must be separable: $n = f(z)g(r)$. Making these substitutions in (2.4), one obtains the fundamental equation

$$4\pi c^2 q(n_i Z - n) + 2\delta B\omega + \frac{1}{q}\left[\frac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)^{\bullet}\right]^{\bullet} + \\ + \frac{1}{qr}\left[r\,\frac{\mathrm{d}p_e}{\mathrm{d}n}(\log n)'\right]' + \frac{mv}{q}\,\Omega^{-1}\,\frac{1}{r}\left(\frac{1}{g}\right)' = 0 \, , \tag{2.8}$$

where $n = f(z)g(r)$. Any separable solution of the n equation leads to a solution for v_r via (2.5), hence to a solution for v_θ by (2.3b). Then E_r and E_z are given directly by the motion eq. (2.3a) and (2.3c). This $\boldsymbol{E}$ is automatically a solution of the Maxwell eq. (2.3c) since the n equation is equivalent to c in virtue of a and c. A second solution for $\boldsymbol{E}$ is given by directly solving (2.3c) with this solution n and the proper boundary conditions. If this n is a *physical* solution, these two solutions for $\boldsymbol{E}$ must agree, at least in some domain. A comparison of them then determines the integration constants in n in terms of $\delta B\omega$ and the geometry of the disc (the ratio ε/R).

Eq. (2.8) very probably has no separable solutions except the trivially separable cases that n is a function of r alone or of z alone. The case $n = n(r)$ must be rejected immediately, since the uniform (in z) charge distribution at any point would certainly engender a non-zero [1] E_z field within the disc which, however, would contradict $E_z = 0$ following from the motion eq. (2.3c). What happens is that the electron distribution takes up that non uniform shape in the z direction in which the electron pressure-z-gradient just balances the

[1] Note that the electrically neutral situation $n(\boldsymbol{r}) = n_i Z = \text{const}$ is *no* solution of (2.8), hence $E_z \not\equiv 0$.

electrostatic field E_z caused by the departure from electrical neutrality (5). Hence we look for a solution $n = n(z)$. Such a solution can only be approximate: we shall discover its domain of validity later. The n equation becomes

$$4\pi c^2 q^2 (n_i Z - n) + 2\delta B\omega q + \left[\frac{dp_e}{dn}(\log n)^{\bullet}\right]^{\bullet} = 0 . \tag{2.9}$$

Introduce the notation

$$N \equiv n_i Z - n$$

for the « ion excess ». [N.B.: although it is true that the relative departure from electrical neutrality is completely negligible in the sense that $|N/n_i Z| \ll 1$, $N = 0$ is *not* a good approximate solution here. By analogy with the current density, all fields here will have *conductive* parts (which depend on n standing alone in a product of factors) and *convective* parts (which depend on n in the combination N). These two parts can be of the same order under certain conditions. n can be replaced by $n_i Z$ with negligible error in the conductive terms, but the same substitution, *i.e.* $N = 0$ in the convective terms is not permissible since it turns out that N varies enormously through the disc.] For simplicity, approximate the electron pressure by the first term on the right in (2.2): $p_e \equiv a n^{\frac{5}{3}}$. Then

$$\left[\frac{dp_e}{dn}(\log n)^{\bullet}\right]^{\bullet} = \frac{5}{2} a (n^{\frac{2}{3}})^{\bullet\bullet} .$$

Write $n = n_i Z(1 - N/n_i Z)$ and assume $|N| \ll n_i Z$. Hence

$$n^{\frac{2}{3}} \approx (n_i Z)^{\frac{2}{3}} [1 - \tfrac{2}{3}(N/n_i Z)]$$

and the n equation becomes the following linear equation for N:

$$\ddot{N} - \varkappa^2 N - \varkappa^2 b = 0$$

where

$$\varkappa \equiv \left[\frac{12\pi c^2 q^2 (n_i Z)^{\frac{1}{3}}}{5a}\right]^{\frac{1}{2}} = 2\left(\frac{3}{\pi}\right)^{\frac{1}{6}} \left[\frac{(n_i Z)^{\frac{1}{3}}}{a_0}\right]^{\frac{1}{2}}, \tag{2.10}$$

and

$$b \equiv \frac{\delta B\omega}{2\pi c^2 q} = \delta \frac{\omega\omega_c}{2\pi c^2 r_0} . \tag{2.11}$$

(5) Throughout this paper we limit ourselves to the region well away from the edges of the metal, hence neglect surface forces and other edge effects.

Here $a_0 = \hbar^2/mc^2q^2 \approx 5.3 \cdot 10^{-9}$ cm is the first Bohr radius and $r_0 \equiv q^2/m \approx$ $\approx 2.8 \cdot 10^{-13}$ cm is the electron classical radius. The general solution is

$$N = (N_0 + b)\cosh \varkappa z - b + d \sinh \varkappa z \qquad [N_0 \equiv N(0)]. \tag{2.12}$$

With this solution for n, the motion equations yield one solution, call it $\boldsymbol{E}^{(1)}$, for the convective part of $\boldsymbol{E}$:

$$\begin{cases} E_r^{(1)} = -\delta B \omega r\,, \\ E_z^{(1)} = -\dfrac{5}{2}\dfrac{a}{q}(n^{\frac{2}{3}})^{\bullet} \approx \dfrac{4\pi e^2 q}{\varkappa^2}\dot{N}(z)\,. \end{cases} \tag{2.13}$$

The direct solution of the Maxwell eq. (2.3e) gives [6] the convective part, say $\boldsymbol{E}^{(2)}$ as

$$\begin{cases} E_r^{(2)} = 2\pi e^2 q R \displaystyle\int_0^\infty \mathrm{d}\lambda J_1(\lambda R)\, J_1(\lambda r) \int_{-\varepsilon/2}^{\varepsilon/2} \mathrm{d}z' \exp\left[-\lambda |z - z'|\right] N(z')\,, \\ E_z^{(2)} = 2\pi e^2 q R \displaystyle\int_0^\infty \mathrm{d}\lambda\, J_1(\lambda R) J_0(\lambda r) \int_{-\varepsilon/2}^{\varepsilon/2} \mathrm{d}z'\, \mathrm{Sgn}(z - z') \exp\left[-\lambda |z - z'|\right] N(z')\,. \end{cases} \tag{2.14}$$

We shall need also the z derivatives of these fields:

$$\dot{E}_r^{(2)} = -2\pi e^2 q R \int \mathrm{d}\lambda\, \lambda J_1(\lambda R) J_1(\lambda r) \int \mathrm{d}z'\, \mathrm{Sgn}(z - z') \exp\left[-\lambda |z - z'|\right] N(z')$$

$$\dot{E}_z^{(2)} = -2\pi e^2 q R \int \mathrm{d}\lambda\, \lambda J_1(\lambda R) J_0(\lambda r) \int \mathrm{d}z' \exp\left[-\lambda |z - z'|\right] N(z') + 4\pi e^2 q\, N(z)\,.$$

The question is now: can these two solutions for $\boldsymbol{E}$ be made to agree in some domain by proper choice of the integration constants?

$E_z^{(1)}$ is no function of r and $E_r^{(1)}$ is no function of z, whereas $\boldsymbol{E}^{(2)}$ involves both co-ordinates. Hence the domain must be restricted to the neighborhood of some point. We shall choose this point to be the origin and determine the corresponding values of the constants. Now the even part N_+ of N leads to the parts $(E_r^{(a)})_+$, $(E_z^{(a)})_-$, $(\dot{E}_r^{(a)})_-$ and $(\dot{E}_z^{(a)})_+$ for $a = 1, 2$, where $+$ and $-$ denote even and odd in z, respectively [7]. Similarly the odd part N_- of N leads to the parts $(E_r^{(a)})_-$, $(E_z^{(a)})_+$, $(\dot{E}_r^{(a)})_+$, and $(\dot{E}_z^{(a)})_-$. To secure agreement in the « linear » neighborhood of the origin (terms $O(z^2)$ and $O(r^2)$ neglected), we must make

[6] Fuller details of the following calculations are given in the Appendix.

[7] Some of these statements are of course vacuously true.

the replacements

$$J_0(\lambda r) \to 1\,, \qquad J_1(\lambda r) \to J_1'(0) r = \tfrac{1}{2}\lambda r$$

and then equate even and odd parts of $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ and of $\dot{\boldsymbol{E}}^{(1)}$ and $\dot{\boldsymbol{E}}^{(2)}$ at $z=0$. The odd parts all vanish at $z=0$, hence we get four equations between even parts which must reduce to identities by the proper choice of the two constants N_0 and d if a physical solution is to exist.

$$(E_z^{(2)})_+ = (E_z^{(1)})_+$$

(at $z=0$ is always hereafter understood) gives

$$2\pi c^2 q R \int \mathrm{d}\lambda\, J_1(\lambda R) \int \mathrm{d}z'\, \mathrm{Sgn}(-z') \exp\left[-\lambda |z'|\right] d \sinh \varkappa z' = d\, \frac{4\pi c^2 q}{\varkappa}\,.$$

The left member of this equation depends on the geometry of the disc, whereas the right member does not (cfr. (2.10)). Therefore $d=0$; $n(z)$ *is even.* $(\dot{E}_r^{(2)})_+ = (\dot{E}_r^{(1)})_+$, which is homogeneous in d, is thus also satisfied by $d=0$.

$(E_r^{(2)})_+ = (E_r^{(1)})_+$ gives

$$2\pi c^2 q r R \int \mathrm{d}\lambda\, \lambda J_1(\lambda R) \int_0^{\varepsilon/2} \mathrm{d}z' \exp\left[-\lambda z'\right]\left[(N_0 + b)\cosh \varkappa z' - b\right] = -\delta B \omega r\,. \tag{2.15}$$

This equation determines N_0 as

$$N_0 = -\delta\, \frac{\omega \omega_c}{2\pi c^2 r_0}\left[1 + \frac{1}{\Theta}\left\{1 - \frac{\varepsilon/2R}{[1+(\varepsilon/2R)^2]^{\frac{1}{2}}}\right\}\right], \tag{2.16}$$

where

$$\Theta \equiv R \int_0^{\infty} \mathrm{d}\lambda\, \lambda J_1(\lambda R) \int_0^{\varepsilon/2} \mathrm{d}z' \exp\left[-\lambda z'\right] \cosh \varkappa z' \approx (\varkappa R)^{-1} \sinh(\varkappa \varepsilon/2)\,.$$

The formula

$$R \int_0^{\infty} \mathrm{d}\lambda\, \lambda J_1(\lambda R) \int_0^{\varepsilon/2} \mathrm{d}z' \exp\left[-\lambda z'\right] = (\varepsilon/2R)\left[1 + (\varepsilon/2R)^2\right]^{-\frac{1}{2}},$$

has been used. Finally $(\dot{E}_z^{(2)})_+ = (\dot{E}_z^{(1)})_+$ gives

$$-4\pi c^2 q R \int \mathrm{d}\lambda\, \lambda J_1(\lambda R) \int_0^{\varepsilon/2} \mathrm{d}z' \exp\left[-\lambda z'\right] N_+(z') + 4\pi c^2 q N_0 = 4\pi c^2 q (N_0 + b)\,.$$

Cancelling the N_0 term and noting the definition of b, we see that this is none other than eq. (2.15) defining N_0. The two solutions are thus consistent in the linear domain.

Domain of validity of the solution: from (2.14) it is seen that for the thin disc ($\varepsilon \ll R$) the main contribution to the λ integral comes when $\lambda R \approx 1.8$, that is, for $\lambda = O(R^{-1})$. Hence expanding the Bessel functions and exponentials, we see that our linearized solution for $\boldsymbol{E}$ is good in the domain

$$(r/R)^2 \ll 1 \qquad \text{when } (\varepsilon/R)^2 \ll 1.$$

The fact that $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ do not agree for large r shows that the solution for n must be restricted to the same r domain. In addition, we have assumed that $|N| \ll n_i Z$. The solution (2.12) would thus break down at a z^* defined by

$$n_i Z = \frac{\omega\omega_c}{2\pi e^2 r_0}\left[1 + \frac{1}{\Theta}\,(1 - \varepsilon/2R)\,\cosh \varkappa z^*\right].$$

However, under normal conditions our eq. (2.12) will be good throughout $(-\varepsilon/2, \varepsilon/2)$. We can see this as follows. Using the approximate evaluation of Θ and the monotonicity of N, the maximum value of N comes out to be (see below, eq. (2.17))

$$N_{\max} = -\delta\,\frac{\omega\omega_c}{2\pi e^2 r_0}\,[1 + \varkappa R(1 - \varepsilon/2R)\,\coth\,(\varkappa\varepsilon/2)]\,.$$

Normally $\varkappa R$ and $\varkappa\varepsilon/2$ are both very large, hence $\coth(\varkappa\varepsilon/2) \approx 1$ and

$$\frac{|N_{\max}|}{n_i Z} \approx \varkappa R \left(\frac{\omega\omega_c}{2\pi e^2 r_0}\right) \Big/ n_i Z\,.$$

Putting in the rather extreme numbers $\omega = 10^3\ \mathrm{s}^{-1}$, $\omega_c = 10^{11}\ \mathrm{s}^{-1}$ for a large disc $R = 10^2$ cm, and using $\varkappa \approx 10^8\ \mathrm{cm}^{-1}$ for $n_i Z = 10^{23}\ \mathrm{cm}^{-3}$, the right member is only about 10^{-8}.

Finally, the laminar flow solution $v_z = 0$ and v_r, v_θ as obtained below depend on the assumption that n is separable, in fact a function of z alone. Hence with the thin disc assumed, they are good also only in the above mentioned r domain.

From (2.7) we get $v_r = \alpha/r$, where the constant α will be determined shortly. Then (2.3b) gives

$$v_\theta = \omega r + \delta\,\frac{\omega_c}{\nu}\,\frac{\alpha}{r}\,.$$

The complete solution for E_r (convective plus conductive part) is then from (2.3a)

$$E_r = -\delta B\omega r - \frac{mv}{q}\,\Omega^{-1}\frac{\alpha}{r}.$$

$N(z)$ is evaluated by substituting $d=0$ and N_0 as given by (2.16) into (2.12); E_z is then completely determined by (2.13). To evaluate α, let I denote the current ([8]) flowing *into* the center of the disc through the center contact, which is thus the current flowing *out* of any little cylinder containing the origin. Then

$$I = -q\,2\pi\int_{-\varepsilon/2}^{\varepsilon/2} \mathrm{d}z\, rnv_r = -2\pi q\varepsilon\alpha\bar{n} \approx -2\pi q\varepsilon\alpha n_i Z\,,$$

which evaluates α. When this value is inserted above, the complete approximate solution is

$$\text{(2.17)}\qquad \begin{cases} N \equiv n_i Z - n = -\delta\,\dfrac{\omega\omega_c}{2\pi v^2 r_0}\left[1+\dfrac{1}{\Theta}\,(1-\varepsilon/2R)\cosh\varkappa z\right], \\ v_r = -\dfrac{I}{2\pi\varepsilon q n_i Z}\cdot\dfrac{1}{r}, \\ v_\theta = \omega r - \delta\,\dfrac{\omega_c}{\nu}\cdot\dfrac{I}{2\pi\varepsilon q n_i Z}\cdot\dfrac{1}{r}, \\ v_z = 0\,, \qquad [(r/R)^2 \ll 1,\ -\varepsilon/2 < z < \varepsilon/2;\quad (\varepsilon/R)^2 \ll 1]\,, \\ E_r = -\delta B\omega r + \Omega^{-1}\,\dfrac{m\nu}{q}\cdot\dfrac{I}{2\pi\varepsilon q n_i Z}\cdot\dfrac{1}{r}, \\ E_\theta = 0\,, \\ E_z = -\delta\,\dfrac{2q}{\varkappa}\,\dfrac{\omega\omega_c}{r_0}\,\dfrac{1}{\Theta}\,(1-\varepsilon/2R)\sinh\varkappa z\,. \end{cases}$$

In the unusual case that $z^* \approx \varepsilon/2$, the restriction $0 \leqslant |z| \ll z^*$ must be added.

Perhaps the main interest of the convective part of this approximate solution lies in the formula for N, indicating the sign and order of magnitude of the departure from electrical neutrality caused by the magnetic field. Notice that N never changes sign and in fact increases monotonically in absolute value

([8]) I here plays the role of another adjustable external parameter, like B. One expects $I=KB$, where the proportionality constant K is practically independent of B and depends on the characteristics of the disc and of the external circuit. In fact one can estimate crudely $K \approx \omega R^2/2R_{\mathrm{ext}}$, where R_{ext} is the resistance of the external circuit.

as $|z|$ goes from 0 to $\varepsilon/2$. Thus there is not only violation of neutrality in the small, but also an overall violation around the center of the disc. Presumably this defect or excess of electrons would be absorbed or supplied, respectively, by the outer portions of the disc [9]. The following correlation exists:

$$\boldsymbol{B} \text{ parallel to } \boldsymbol{\omega}(\delta = +1) \Leftrightarrow \text{electron excess}$$

$$\boldsymbol{B} \text{ antiparallel to } \boldsymbol{\omega}(\delta = -1) \Leftrightarrow \text{electron defect}.$$

In order of magnitude

$$N_0 \approx -\delta \frac{\omega\omega_c}{2\pi c^2 r_0}. \tag{2.18}$$

For $\omega = 10^3\ \mathrm{s}^{-1}$ and $\omega_c = 10^{11}\ \mathrm{s}^{-1}$ ($B \approx 10^4$ gauss) this gives $N_0 = -\delta \cdot 10^5\ \mathrm{cm}^{-3}$, or there is violation of neutrality by some hundred thousand electron charges per cubic centimeter on the median plane of the disc for these rather extreme conditions.

The current density: Knowing $\boldsymbol{v}$, we can now form the total current density

$$\boldsymbol{j} \equiv q(n_i Z \boldsymbol{\omega} \times \boldsymbol{r} - n\boldsymbol{v}).$$

This gives

$$\begin{cases} j_r = \dfrac{I}{2\pi\varepsilon} \cdot \dfrac{1}{r}, \qquad j_z = 0, \\ j_\theta = qN\omega r + \delta \dfrac{\omega_c}{\nu} \dfrac{I}{2\pi\varepsilon} \cdot \dfrac{1}{r}, \end{cases} \tag{2.19}$$

where we have replaced $n/n_i Z$ by unity. The form of the purely conductive j_r could have been inferred from conservation arguments alone, of course, given cylindrical symmetry. The azimuthal component j_θ shows both convective and conductive parts. The current stream lines $\theta = f(r)$ in any plane $z = \text{const.}$ are obtained by integrating the equations

$$r \frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{j_\theta}{j_r}. \tag{2.20}$$

This yields the locus

$$\theta = \frac{\pi\varepsilon q\omega N}{I} r^2 + \delta \frac{\omega_c}{\nu} \log\left(\frac{r}{R_0}\right), \tag{2.21}$$

where R_0 is some fixed length to be determined by suitable boundary conditions for the finite disc. *The streamlines are spirals.* Near enough the origin the logarithmic term predominates: the curve spirals clockwise or counter-

[9] This seems to be the only explanation since this simple theory makes no provision for exchange of electrons with the ambient medium.

clockwise (meaning when one traces the curve in the sense of increasing r) according as $\delta = -1$ or $+1$. Sufficiently far from the origin, the quadratic term is dominant, and the curve spirals clockwise or counterclockwise according as $\mathrm{Sgn}\,(N/I) = -1$ or $+1$. Consideration of the direction of the Lorentz force on the electrons in the disc shows that they tend to be urged inward or outward if $\boldsymbol{B}$ is parallel or antiparallel to $\boldsymbol{\omega}$, respectively. Thus

$$\mathrm{Sgn}\, I = \delta\,.$$

Since $\mathrm{Sgn}\, N = -\delta$, the curve always spirals clockwise far from the origin. The sense is then given to the complete streamlines by noting that the current must flow out from the center or in toward the center according as $\mathrm{Sgn}\, j_r = +1$ or -1; but $\mathrm{Sgn}\, j_r = \mathrm{Sgn}\, I = \delta$.

A critical point where the complete spiral changes sense occurs when $\mathrm{d}\theta/\mathrm{d}r = 0$, or by (2.20), where $j_\theta = 0$. This can happen only in the case of an electron excess ($N < 0$, $\delta = +1$). One obtains

$$(r_{\mathrm{crit}})^2 = \frac{\omega_c}{2\pi\varepsilon q\omega v}\left|\frac{I}{N}\right|\,. \qquad (\delta = +1)\,.$$

This is also the point (in either case $\delta = \pm 1$) at which the convective and conductive parts of j_θ are of the same size. Using footnote [8] to evaluate I roughly in terms of the characteristics of the system and (2.18) to evaluate N crudely on the median plane $z = 0$, we can cast this into the form

$$\left(\frac{r_{\mathrm{crit}}}{R}\right)^2 = \frac{1}{2}\,\frac{\omega_c}{\omega}\,\frac{c^2}{v\varepsilon R_{\mathrm{ext}}}\,.$$

Taking $\omega = 10^3\ \mathrm{s}^{-1}$, $\omega_c = 10^{11}\ \mathrm{s}^{-1}$, $R_{\mathrm{ext}} = 10^{12}$ emu $(= 10^3$ ohm), $\varepsilon = 10^{-1}$ cm, $v = 10^{12}\ \mathrm{s}^{-1}$ this gives $r_{\mathrm{crit}} = 10^3 R$, which means that the critical point does not exist under these conditions. On the other hand, for weak fields, high angular velocities, and/or large external resistances (*e.g.*, $B = 1$ gauss, $\omega = 10^2\ \mathrm{s}^{-1}$, and $R_{\mathrm{ext}} = 10^{16}$ emu, whence $r_{\mathrm{crit}} = R/10$) one would expect to find a critical point.

3. – The Boltzmann equation.

The same problem can be treated from the more fundamental point of view of the Boltzmann transport equation for the velocity distribution function of the electron gas. In this section we solve the Boltzmann equation for the electron distribution function and then use it to calculate the electron diffusion; *i.e.*, the current density, in the Faraday disc. The distribution function

we obtain can, of course, be used to calculate other quantities of interest as well.

In the first approximation the ions and electrons are in equilibrium at temperature T;

$$f^{(0)} = \frac{2m^3}{h^3}\left[1 + A \exp\left(\frac{mC^2}{2kT}\right)\right] \tag{3.1}$$

is the electron distribution function whereas the ions have a Maxwellian distribution appropriate to a non-degenerate gas. Notation: let $\boldsymbol{c}_0$ be the mean velocity of the equilibrium binary mixture, by hypothesis

$$\boldsymbol{c}_0 = \boldsymbol{\omega} \times \boldsymbol{r} \, ;$$

let $\boldsymbol{c}$ denote electron velocity; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{c}_0$ is the electron velocity relative to the centroid. The condition for considerable degeneracy is [10]

$$h^3 n / 2(2\pi m\, kT)^{\frac{3}{2}} \gg 1 \, ,$$

where n is, as before, the electron number density; when this is true, $A \ll 1$ in (3.1). In the second approximation we write $f^{(1)} = f^{(0)}(1 + \Phi^{(1)})$; then the Boltzmann equation for the electrons becomes [11]

$$\boldsymbol{C} \cdot \nabla f^{(0)} - \frac{q}{m}(\boldsymbol{E} + \boldsymbol{c}_0 \times \boldsymbol{B}) \cdot \frac{\partial f^{(0)}}{\partial \boldsymbol{C}} - \frac{\partial f^{(0)}}{\partial \boldsymbol{C}} \boldsymbol{C} : \nabla \boldsymbol{c}_0 =$$

$$- \frac{q}{m} f^{(0)}(\boldsymbol{C} \times \boldsymbol{B}) \cdot \frac{\partial \Phi^{(1)}}{\partial \boldsymbol{C}} + \int f_i^{(0)} f^{(0)}(\Phi^{(1)\prime} - \Phi^{(1)}) \varkappa_{12}(C, \chi) \sin \chi \, \mathrm{d}\chi \, \mathrm{d}\varepsilon \, \mathrm{d}^3 C \, .$$

Here we have used some special features of our problem, namely

$$\frac{D}{Dt} f^{(0)} = \frac{D}{Dt} \boldsymbol{c}_0 = 0 \, , \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{c}_0 \cdot \nabla \, .$$

The Lorentzian approximation [12] has been used to write the collision term; this approximation should be especially good here, where the ions not only much outweigh the electrons, but are tightly bound into a lattice as well. $f_i^{(0)}$ is the equilibrium ion distribution. The CC notation for an inner product of dyads has been used:

$$\frac{\partial f^{(0)}}{\partial \boldsymbol{C}} \boldsymbol{C} : \nabla \boldsymbol{c}_0 \equiv \sum_{i,j} \frac{\partial f^{(0)}}{\partial C^i} C^j \frac{\partial c_0^i}{\partial x^j} \, ,$$

---

[10] At $T = 300$ °K this quantity ≈ 3000 for copper.

[11] CC (18.4$_6$).

[12] CC, Sect. 10.5.

the notation is otherwise as in CC. This equation is solved in the manner expounded so systematically in CC. The result is

$$\Phi^{(1)} = \frac{Q_1}{B}\frac{\omega_c}{\nu}\left(\boldsymbol{E} + \boldsymbol{c}_0\times\boldsymbol{B} - \frac{m}{qQ_1}\nabla\log f^{(0)}\right)\cdot \tag{3.3}$$

$$\cdot\left[\boldsymbol{C} + \Omega\frac{\omega_c}{\nu}\frac{1}{B}\boldsymbol{C}\times\boldsymbol{B} + \Omega\left(\frac{\omega_c}{\nu}\right)^2\frac{1}{B^2}(\boldsymbol{C}\times\boldsymbol{B})\times\boldsymbol{B}\right] +$$

$$+ \frac{Q_1}{4\nu'}\nabla\boldsymbol{c}_0 : \left[\boldsymbol{C}\boldsymbol{C} + \Psi\left(\boldsymbol{C} + \frac{\omega_c}{\nu}\frac{1}{B}\boldsymbol{C}\times\boldsymbol{B}\right)\left(\boldsymbol{C} + \frac{\omega_c}{\nu}\frac{1}{B}\boldsymbol{C}\times\boldsymbol{B}\right)\right].$$

Here ν and ν' are collision frequencies, functions of C, of the first and second kinds defined by

$$\begin{cases} \nu \equiv 2\pi n_i \int (1-\cos\chi)\alpha_{12}(C,\chi)\sin\chi\,\mathrm{d}\chi\,, \\ \nu' \equiv \dfrac{3}{2}\pi n_i \int (1-\cos^2\chi)\alpha_{12}(C,\chi)\sin\chi\,\mathrm{d}\chi\,, \end{cases} \tag{3.4}$$

where $n_i \equiv$ ion number density as before, and

$$\Omega = \left[1 + \left(\frac{\omega_c}{\nu}\right)^2\right]^{-1}, \qquad \Psi = \left[1 + \left(\frac{\omega_c}{\nu'}\right)^2\right]^{-1}.$$

Also

$$Q_1 \equiv \frac{1}{C}\frac{\partial}{\partial C}\log f^{(0)} = -\frac{Am}{kT}\exp\left[\frac{mC^2}{2kT}\right]\left(1 + A\exp\left[\frac{mC^2}{2kT}\right]\right)^{-1}.$$

When (3.3) is specialized to the case $\boldsymbol{c}_0 = 0$ (metal at rest), and purely transverse $\boldsymbol{E}$ and density gradient, it reduces to Sommerfeld's solution ([13]), where his mean free path l is connected with our collision frequency ν by $\nu = C/l$. The terms involving $\boldsymbol{c}_0$ in (3.3) give the effect of the « dynamo action » on the electron gas. [The absence of terms involving $(\boldsymbol{C}\times\boldsymbol{B})\times\boldsymbol{B}$ in the quadratic part of (3.3) is due to the special feature of our problem that the dyad $\nabla\boldsymbol{c}_0$ has no longitudinal components.]

The total current density splits conveniently into a convective and a conductive part:

$$\boldsymbol{j} \equiv q(n_i Z\overline{\boldsymbol{c}}_i - n\overline{\boldsymbol{c}}) \approx q(n_i Z - n)\boldsymbol{c}_0 - nq\,\overline{\boldsymbol{C}},$$

([13]) CC (18.5$_4$).

where we have used the approximation, excellent for metallic ions, $\bar{\boldsymbol{c}}_i = \boldsymbol{c}_0$. The conductive part

$$\boldsymbol{j}_{\text{cond}} \equiv -nq\bar{\boldsymbol{C}} \equiv -q\int f^{(0)}\Phi^{(1)}\boldsymbol{C}\,\mathrm{d}^3C\,,$$

is calculated as in CC. The term involving the quadratic expressions in $\boldsymbol{C}$ in (3.3) thereby contribute nothing by symmetry. This gives

$$\begin{aligned}(3.5)\qquad \boldsymbol{j}_{\text{cond}} = \frac{4\pi}{3}q\Bigg\{&\nabla\int\frac{C^4}{\nu}f^{(0)}\mathrm{d}C + \frac{\omega_c}{B}\boldsymbol{B}\times\nabla\int\frac{\Omega C^4}{\nu^2}f^{(0)}\mathrm{d}C + \\ &+\left(\frac{\omega_c}{B}\right)^2\boldsymbol{B}\times\left[\boldsymbol{B}\times\nabla\int\frac{\Omega C^4}{\nu^3}f^{(0)}\mathrm{d}C\right] + \frac{\omega_c}{B}\int\frac{\partial}{\partial C}\left[\frac{C^3}{\nu}\left\{\boldsymbol{E}+\boldsymbol{c}_0\times\boldsymbol{B}+\right.\right. \\ &\left.\left.+\Omega\frac{\omega_c}{\nu}\frac{1}{B}\boldsymbol{B}\times(\boldsymbol{E}+\boldsymbol{c}_0\times\boldsymbol{B})+\Omega\left(\frac{\omega_c}{\nu}\right)^2\frac{1}{B^2}\boldsymbol{B}\times[\boldsymbol{B}\times(\boldsymbol{E}+\boldsymbol{c}_0\times\boldsymbol{B})]\right\}\right]f^{(0)}\mathrm{d}C\Bigg\},\end{aligned}$$

where $f^{(0)}$ is defined in (3.1). The integrations can be done using the formula

$$\int_0^\infty\frac{f(x)\,\mathrm{d}x}{1+A\exp[x]} \approx \int_0^{x_0} f(x)\,\mathrm{d}x\,, \qquad (A \ll 1),$$

with $x_0 = \log(1/A)$; this corresponds to using the crudest approximation [14] for the electron distribution in which $f^{(0)} = 2m^3h^3$ for $C \leqslant w$, $f^{(0)} = 0$ for $C > w$ where w is the particular high velocity defined by

$$(3.6)\qquad \frac{mw^2}{2kT} = \log\left(\frac{1}{A}\right).$$

In this same approximation w is given in terms of the electron density by

$$(3.7)\qquad w = \left(\frac{3}{8\pi}\frac{h^3}{m^3}n\right)^{\frac{1}{3}}.$$

One obtains in this approximation

$$(3.8)\qquad \begin{cases}\boldsymbol{j}_{\text{cond}} = \dfrac{nq^2}{m\nu(w)}\left[\boldsymbol{F}+\Omega(w)\dfrac{\omega_c}{\nu(w)}\dfrac{1}{B}\boldsymbol{B}\times\boldsymbol{F}+\Omega(w)\left[\dfrac{\omega_c}{\nu(w)}\right]^2\dfrac{1}{B^2}\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{F})\right]; \\ \boldsymbol{F} \equiv \boldsymbol{E}+\boldsymbol{c}_0\times\boldsymbol{B}+\dfrac{1}{qn}\nabla p_e\,.\end{cases}$$

[14] It seems to us that the refinements in the theory of the current density which would be obtained by doing the integrals in (3.5) more accurately (which can easily be done) would be of considerable interest.

Formula (3.7) has been used to express the last term of $\boldsymbol{F}$ in terms of the electron pressure according to

$$\frac{1}{3}\frac{m}{q}\frac{w^2}{n}\nabla n = \frac{1}{qn}\nabla p_e\,, \tag{3.9}$$

where p_e to this approximation is just the first term on the right in (2.2). In $\boldsymbol{j}_{\text{cond}}$ the functions of C, Ω and ν, are evaluated at $C = w$.

The generalized Ohm's Law Sp (2-12), when solved for $\boldsymbol{j}$ in the steady case, yields ([15])

$$\boldsymbol{j} = \frac{nq^2}{m\nu}\left[\boldsymbol{F} + \Omega\frac{\omega_c}{\nu}\frac{1}{B}\boldsymbol{B}\times\boldsymbol{F} + \Omega\left(\frac{\omega_c}{\nu}\right)^2\frac{1}{B^2}\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{F})\right],$$

for the same $\boldsymbol{F}$ as above. We have replaced the resistivity η by the simple expression (1.4). As explained in the Introduction, this current density must be interpreted as the conductive part. It is identical in form with (3.8). The more fundamental derivation thus indicates that the collision frequency ν, which occurs as a sort of vaguely defined average in the elementary theory, should be that velocity dependent function defined by the integral (3.4) evaluated at the specific large velocity w. The largeness of this « average velocity » is a manifestation of the advanced degeneracy.

The more fundamental theory has finally led us to the expression (3.8) for the conduction current. If we now set $\boldsymbol{c}_0 = \boldsymbol{\omega}\times\boldsymbol{r}$ and $\overline{\boldsymbol{c}} = \boldsymbol{v}$ as in Section 2,

$$\boldsymbol{j}_{\text{cond}} = -nq\overline{\boldsymbol{C}} = -nq(\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{r})$$

and it is easily shown that the r, θ, and z components of (3.8) give us just the first three eq. (2.3). Also since $\nabla\cdot\boldsymbol{j}_{\text{conv}} = \nabla\cdot[(n_iZ - n)(\boldsymbol{\omega}\times\boldsymbol{r})] = 0$, the continuity equation for $\boldsymbol{j}$ reads

$$0 = \nabla\cdot\boldsymbol{j}_{\text{cond}} = -q\nabla\cdot(n\boldsymbol{v})$$

which just furnishes the fourth eq. (2.3). The two Maxwell equations for $\boldsymbol{E}$ supplement and complete this set. We are led back to the same set of equations from which we started in the elementary theory. The end result is to replace ν and Ω in the solutions (2.17) by the well defined $\nu(w)$ and $\Omega(w)$.

([15]) Interesting point: resolving $\boldsymbol{j}$ into longitudinal $j_\parallel$ and transverse $j_\perp$ parts, one sees that $j_\perp$ is provided with the magnetic shrinking factor Ω, but $j_\parallel$ is not; because $(1/B)^2[\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{F})]_\perp = -F_\perp$ but $(1/B)^2[\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{F})]_\parallel = 0$.

Appendix

The solution for the electrostatic potential is

$$\varphi(\boldsymbol{r}) = c^{-}q\int r'\,\mathrm{d}r'\,\mathrm{d}z'\,\mathrm{d}\varphi'\,\frac{N(z')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,. \tag{A.1}$$

The expansion

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{m=-\infty}^{\infty}\exp\left[im(q-q')\right]\int_0^{\infty}\mathrm{d}\lambda J_m(\lambda r)J_m(\lambda r')\exp\left[-\lambda|z-z'|\right], \tag{A.2}$$

convenient in cases of cylindrical symmetry, is substituted into (A.1). By cylindrical symmetry, the angle integral of the infinite series collapses to $2\pi\times$ the single term $m=0$. $N(z')$ is also independent of r', hence the r' integral can be done. This gives

$$\varphi = 2\pi c^2 qR\int\mathrm{d}\lambda\lambda^{-1}J_1(\lambda R)J_0(\lambda r)\int\mathrm{d}z'\exp\left[-\lambda|z-z'|\right]N(z')\,. \tag{A.3}$$

Differentiation of this formula yields (2.14) and (2.15) if one uses the formulas

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}z}\exp\left[-\lambda|z-z'|\right] = -\lambda\,\mathrm{Sgn}\,(z-z')\exp\left[-\lambda|z-z'|\right],\\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{Sgn}\,(z-z') = 2\delta(z-z')\,.\end{cases} \tag{A.4}$$

RIASSUNTO (*)

Si studia la distribuzione della corrente in un disco di Faraday (generatore omopolare), con una soluzione approssimata delle equazioni magnetoidrodinamiche. Ne risulta che le linee di flusso sono spirali, con possibilità di cambio del verso nel caso in cui il campo magnetico e l'asse di rotazione siano paralleli. Una notevole caratteristica è l'assenza di soluzioni che si presentino elettricamente neutre. La densità della carica indotta è proporzionale a $B\omega$ (B campo magnetico applicato, ω = velocità angolare del disco). Poi, con un trattamento più completo, si risolve l'equazione di Boltzmann per il disco di Faraday. Ciò estende al caso di metalli in moto il lavoro di Sommerfeld sui metalli in un campo magnetico.

(*) *Traduzione a cura della Redazione.*

Production of Atmospheric Neutrons by Solar γ Rays (*).

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(ricevuto il 4 Ottobre 1960)

This brief note is intended to show the contribution of high energy solar γ radiation, if any exists, to the peculiar phenomenon of neutron intensity transition in the upper atmosphere which was found by HAYMES [1] during his sunset balloon experiments.

The estimation of this contribution is based on the following considerations. During the time of sunset, the necessary penetration depth of solar γ rays to the balloon location increases rapidly. Therefore, if the energy of the γ's is higher than the critical energy of the electron cascade in the air, the number of photons and electrons increases by cascade processes. Since the cross section of photoneutron production of air nuclei has a sharp maximum between 20 and 30 MeV, the rate of neutron production will increase with increase of secondary γ rays in the air.

The corresponding neutron intensity measured by slow neutron detectors can be estimated by considering the diffusion of slow neutrons in the air, which is actually simplified by making use of Fermi's age-factor.

The rate of photoneutron production in the air can be evaluated by making use of the integral cross-section, σ_{int} defined by

$$\sigma_{\text{int}} = \int_{W_{\text{min}}}^{W_0} \sigma(W)\,\mathrm{d}W\,, \tag{1}$$

where the threshold energy, W_{min}, is 10.7 MeV for N_2 and 16.3 MeV for O_2 [2].

As shown by STRAUCH [3], $\sigma(W)$ has a maximum for some γ ray energy, W_e, the effective energy for photoneutron production. According to the measurements by PERLMAN and FRIEDLANDER [4], the cross-section for ${}^{16}_{8}\mathrm{O}(\gamma, \mathrm{n}){}^{15}_{8}\mathrm{O}$ and ${}^{14}_{7}\mathrm{N}(\gamma, \mathrm{n}){}^{13}_{7}\mathrm{N}$ is of order 0.1 barn and $5 \cdot 10^{-2}$ barn, respectively. On the other hand, GAERTTNER and YEATER [5] showed that W_e for the (γ, n) reaction in N_2 and O_2 is of the order of 30 MeV,

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(1) R. C. HAYMES: *Phys. Rev.*, **116**, 1231 (1959).

(2) B. ROSSI: *High Energy Particles* (New York, 1952).

(3) K. STRAUCH: *Phys. Rev.*, **81**, 973 (1951).

(4) M. L. PERLMAN and G. FRIEDLANDER: *Phys. Rev.*, **74**, 446 (1948).

(5) E. R. GAERTTNER and M. L. YEATER: *Phys. Rev.*, **77**, 714 (1950).

and that σ_{int} is approximately 0.4 MeV-barn for both reactions.

In the present calculation the integral cross section of photoneutron production in the air, σ_{int} is assumed to be 0.5 MeV-barn, considering the following points: 1) For high energy γ rays (> 100 MeV) there is additional photoneutron production according to the reaction [6]

$$^{14}_{7}\mathrm{N}+\gamma \rightarrow {}^{1}_{1}\mathrm{H}+3{}^{4}_{2}\mathrm{He}+{}^{1}_{0}\mathrm{n}\,,$$

and 2) nitrogen is four times as abundant as oxygen in the air.

where $\sigma_{int} = 0.5$ MeV-barn,

$$W_e = 20 \text{ MeV} \quad \text{and} \quad N(x) = N\frac{\varrho(x)}{\varrho_0}\,.$$

N is the Loschmidt number ($2.7\cdot10^{19}$/cm^3), and $\varrho(x)$ and ϱ_0 are the air densities at depth x and at NTP, respectively.

The differential intensity of γ rays $\gamma(W_0, W, t)$ at depth t in radiation units corresponding to x in $g\,\mathrm{cm}^{-2}$, with energy between W and $W+\mathrm{d}W$, produced by an incident primary γ ray of

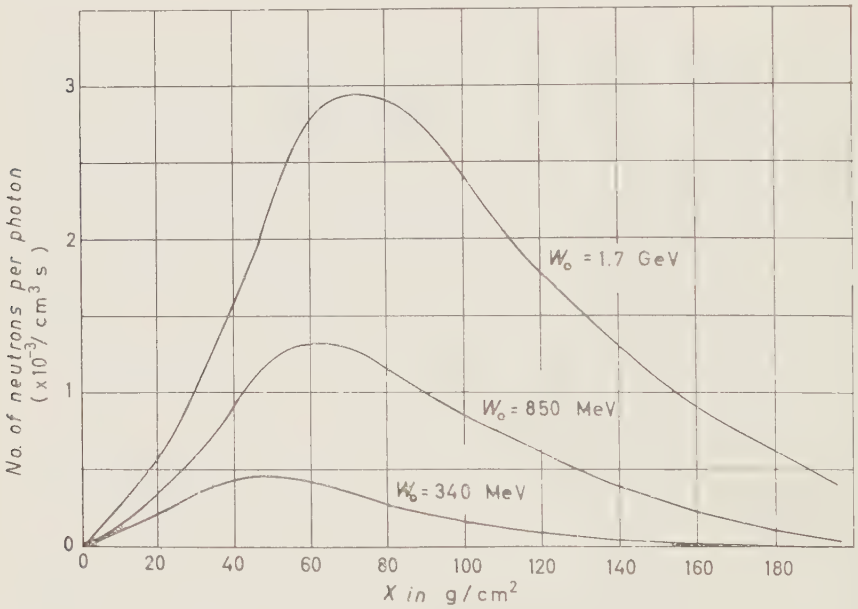

Fig. 1. – Neutron yield in air for various energies of incident photons.

The intensity of neutrons produced at the penetration depth, x (in g/cm^2), by the secondary γ rays is given by

$$n(x, W_0) = \int_{W_{min}}^{W_0} N(x)\sigma(W)\gamma(W_0, W, x)\,\mathrm{d}W \simeq$$
$$\simeq N(x)\gamma(W_0, W, x)\sigma_{int}\,,$$

energy W_0 can be found by the method of Approximation B of the electron-cascade theory [7,8].

In Fig. 1, $n(x, W_0)$ is shown for $W_0 = 340$, 850 and 1700 MeV.

The intensity of slow neutrons, $I(x)$, observed by BF_3 counters at the pene-

[6] G. C. Baldwin and G. S. Klaiber: *Phys. Rev.*, **73**, 1156 (1948).

[7] B. Rossi and K. Greisen: *Rev. Mod. Phys.*, **13**, 240 (1941).

[8] L. Eyges: *Phys. Rev.*, **81**, 981 (1951).

tration depth x along the path of solar γ rays is given approximately by

$$(3) \qquad I(x) = \frac{1}{\sqrt{2\pi r}} \int_0^\infty \exp\left[-\frac{(x-x')^2}{\alpha\tau}\right] \cdot n(x', W_0)\mathrm{d}x',$$

The $I(x)$ curves are normalized at $x = 0$.

From Fig. 2, one can see that (*a*) the observed maximum intensity of slow neutrons, that is roughly 100% above the average, can be explained by 300 MeV solar γ rays, if they exist, but (*b*) the atmospheric depth corresponding to the observed transition maximum, 140 g/cm^{-2},

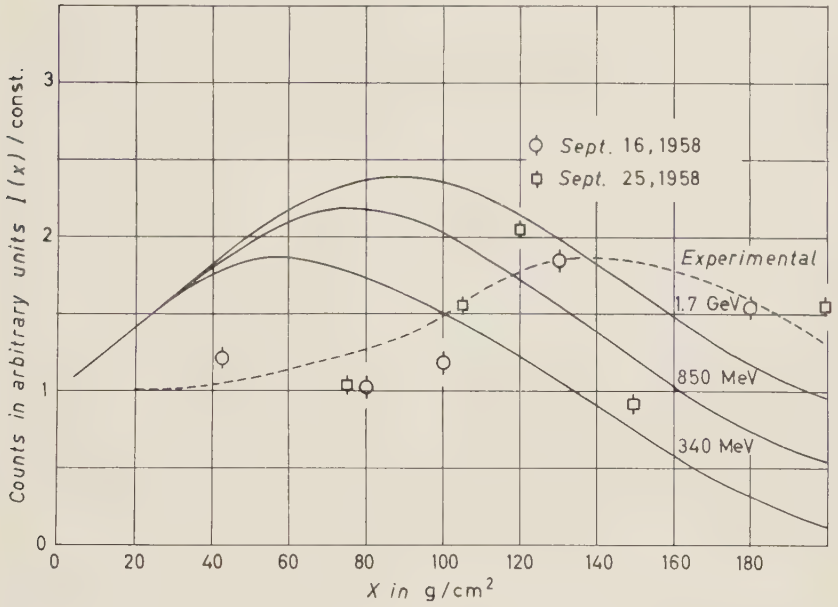

Fig. 2. – Relative intensity of photoneutrons produced in the upper atmosphere (calculated) and the observed transition curve found in the sunset balloon experiment (dotted line).

where the source function $n(x, W_0)$ is given by eq. (2) and shown in Fig. 1. The Fermi-age, δ, is shown as a function of the kinetic energy of the neutron by FUJIMOTO and TAMURA (9). In Fig. 2, $I(x)$ is shown for different energies of incident γ rays in comparison with the transient curves of sunset balloon experiments (shown by the dotted line) (*).

is difficult to ascribe to the γ ray cascade, unless the energy of primary γ rays exceeds several GeV.

Since the variation of penetration depth of solar radiation to the balloon altitude is very fast at the time of large zenith angle of the sun, one cannot yet assume that there is a discrepancy between the observed depth of transient maximum and that calculated for photoneutrons produced by solar γ rays energies of several hundreds MeV.

In this respect, further repetition of this type of experiments at sunrise as well as at sunset, coupled with direct measurements of high energy solar γ rays,

(9) Y. FUJIMOTO and T. TAMURA: *Prog. Theor. Phys.*, **8**, 221 (1952).

(*) By private communication with the author (1), the zenith angle of the sun corresponding to the maximum intensity of neutrons at the balloon height is corrected to 84° from its original 90°

TABLE I. – *Flare activity during sunset balloon experiments.*

Time of balloon flight (U.T.)		Duration of optical solar flares
September 16-17, 1958		September 16, 1958
Ascension	22:40	Importance 1, 22:27 – 23:20
Ceiling	00:05	Importance 1, 22:50 – 23:00
Sunset	01:10	
September 25-26, 1958		September 25, 1958
Ascension	22:10	Importance 1, 22:58 – 23:20
Ceiling	23:00	
Sunset	00:40	

seems necessary. The latter has been recently attempted by DANIELSON ([10]) by means of thin-wall G-M counters. Although his result shows no significant difference between off-sun and on-sun direction, the value of 10^{-2} photons/cm^2 s as an upper limit of the high energy (> 200 MeV) solar γ ray flux, can be regarded as the intensity of γ radiation from the *quiet* sun, because no flare is reported during his measurements.

It is worth-while to note that during the sunset balloon experiments, which have been done only twice so far, several coincident solar flares occurred, as are listed in the Table I ([11]).

* * *

I wish to express my appreciation to Prof. S. A. KORFF, New York University, and Prof. R. L. CHASSON, who encouraged me to work on this problem.

([10]) R. E. DANIELSON: *Journ. Geophys. Res.*, **65**, 2055 (1960).

([11]) Preliminary Report of Solar Activity, High Altitude Observatory, Bulder, (Colorado, 1958).

Muonium-Antimuonium Transitions.

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(ricevuto il 27 Dicembre 1960)

Such decays as $\mu \to e+\gamma$, $\mu \to e+e+\bar{e}$, and $K \to \pi+\mu+\bar{e}$ have never been seen. Should a quantum number, m, be attributed to charged leptons so that $m(\mu^{\pm}) = -m(e^{\pm}) = \pm 1$, the absence of these modes indicates a selection rule, $\Delta m \neq 2$. The muon decay violates this selection rule unless there exist two distinct two-component neutrinos. In that case, it is characterized by $\Delta m = 0, 2, 4$ depending on the assignment of m to the neutrinos. Before it may be asserted that m is conserved, one must determine whether or not there occur phenomena with $\Delta m = 4$. The interaction,

$$g^2 m_\mu^{-2} (\bar{\mu}\gamma_\lambda e)(\bar{\mu}\gamma_\lambda e) + \text{H.c.},$$

leads to the $\Delta m = 4$ processes: $e^- + e^+ \to \mu^- + \mu^-$ and $\mu^+ + e^- \to \mu^- + e^+$. Present experimental evidence regarding this, or a similar, interaction seems scanty. If $\Delta m = 4$ processes are observed, m cannot be a good quantum number; the absence of $\Delta m = 2$ processes may then be attributed to a « multiplicative conservation law » for m [1], or alternatively, it may be said that m is conserved modulo four.

The first mentioned process requires a c.m. energy of the electrons in excess of two muon masses. This will soon be available at the Stanford Colliding Beam experiment. At the same momentum transfer, k^2, the cross-section for this process, compared with the cross-section of the purely electromagnetic process $e^+ + e^- \to \mu^+ + \mu^-$, is approximately

$$(137 g^2/4\pi)^2 (k/m_\mu)^4 .$$

It seems reasonable to suppose that the Stanford experiment will be able to detect $g^2/4\pi > 10^{-4}$.

The second process could be studied by looking for charge-exchange scattering of μ^+ on electrons. This experiment has the convenience that it may be done at the same time that elastic electromagnetic scattering is examined. At present, an experiment of Deery and Neddermayer [2] establishes the limit $g^2/4\pi \lesssim 1/137$.

The transition of muonium into antimuonium by means of the $\Delta m = 4$ coupling was first discussed by Pontecorvo[3]. These transitions would lead to spectacular appearances of μ^- capture stars or to energetic electrons following muonium formation. This is especially of interest now that the muonium system has been

[1] R. Gatto: *Proc. Conf. on High Energy Phys. at Rochester* (1960), p. 609.

[2] R. Deery: private communication; see also, R. Deery and D. Neddermayer: *Phys. Rev.*, in press.

[3] B. Pontecorvo: *Žurn. Eksp. Teor. Fiz.*, **33**, 549 (1957).

produced ([4]). In the presence of certain $\Delta m=4$ interactions (such as the one we suggest), muonium and antimuonium are not degenerate energy eigenstates but can make transitions into one another ([5]). Equivalently, the symmetric and antisymmetric combinations of muonium and antimuonium develop an energy difference, $\Delta E=g^2 m_\mu^{-2}\varphi^2(0)$, where $\varphi(0)$ is the value of the electron wave function at the muon in the ground state of muonium. Corresponding to this is the rate of transition between muonium and antimuonium,

$$1/\tau_g = \Delta E = g^2/\pi \left(\frac{m_e}{m_\mu}\right)^2 (137)^2 \, m_e \, .$$

(in units where $\hbar=c=1$). In the absence of collisions, both muonium and antimuonium have the same lifetime, τ_d, very similar to that of a free muon. If pure muonium is isolated, it consists initially of a superposition of the symmetric and antisymmetric eigenstates with equal amplitude. At later times, an admixture of antimuonium develops with amplitude,

$$(1 - \exp[it/\tau_g]) \exp[-t/2\tau_d] \, .$$

It follows that the fraction, $f=\lambda/(2+2\lambda)$, of the systems will exhibit antimuonium decays [$\lambda=(\tau_d/\tau_g)^2$]. If the $\Delta m=4$ coupling is equal in strength to electromagnetism ($g^2=4\pi\alpha$), $\tau_g\approx 10^{-9}$ s and $f=\frac{1}{2}$; if it is comparable to the weak interactions,

$$(g^2\approx 3\cdot 10^{-7}), \ \tau_g\approx 10^{-3} \text{ s and } f\approx 10^{-5} \, .$$

Unfortunately, muonium is not produced in isolation, but within a medium with which it strongly interacts. Pontecorvo noted that collisions have the effect of reducing the efficiency of the muonium system as a detector of $\Delta m=4$ interactions in so far as they serve to lift the degeneracy between muonium and antimuonium. We shall suppose that the principal effect quenching muonium-antimuonium transitions is the annihilative collision of antimuonium (*i.e.*, collisions resulting in either the annihilation of the positron or the nuclear capture of the μ^-, leading to the total disruption of the antimuonium system). This effect is quite independent of the mass splitting between muonium and antimuonium caused by inhomogeneous electromagnetic fields within the medium, so that the actual quenching effect may be even greater than what we find.

Let τ_a be the lifetime of an antimuonium system within the medium. As a result of annihilative collisions, $\tau_d/\tau_a \gg 1$. Assuming also, $\tau_g/\tau_a \gg 1$, we find $f\approx 4(\tau_a/\tau_d)\lambda$. Thus the incidence of antimuonium decays, compared with an isolated system, is suppressed by the factor $8\tau_a/\tau_d$. If p is the probability that a collision of antimuonium is annihilative, this factor is just $8/(Np)$, where N is the number of collisions to which the system is subject in its natural lifetime, τ_d. Transitions of muonium into antimuonium can yield a sensitive measure of the $\Delta m=4$ interaction only if muonium is to be prepared in very dilute media: the detection of $\Delta m=4$ interactions comparable in strength with weak interactions would appear to be a formidable problem.

* * *

A related, but more comprehensive, work by Drs. G. Feinberg and S. Weinberg was presented by Dr. Weinberg at the 1960 Winter Meeting of the American Physical Society. We are indebted to these authors for a critical discussion. We wish to thank the Institute for Theoretical Physics at Copenhagen for its kind hospitality while this work was done.

([4]) V. W. Hughes D. W. Mc Colm, K. Ziock and R. Prepost: *Phys. Rev. Lett.*, **5**, 63 (1960).

([5]) Some $\Delta m = 4$ interactions have no diagonal matrix elements connecting muonium with antimuonium; they induce no degenerate transitions in lowest order. See: S. Glashow: *Phys. Rev. Lett.*, **6**, 196 (1961).

Electric and Magnetic Polarizabilities of the Nucleon.

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In connection with experimental determinations of the electric polarizability of the nucleon the expectation value of this polarizability $\alpha=(16\div 18)\cdot 10^{-43}$ cm^3 has been evaluated theoretically [1]. In the cited paper only one diagram of the type *A* (see Fig. 1) has been taken into account, the contribution of the other ones has been approximated by the factor 2. The magnitude of the magnetic polarizability β was not calculated since estimates showed that $\beta \ll \alpha$ (*).

In connection with recently published new experimental data it is interesting to evaluate α and β more precisely.

Calculations showed that

$$(1)\quad \alpha=\frac{4e^2}{3\pi}f^2\int_0^\infty \mathrm{d}k\,\exp[-k^2/a^2]\frac{k^4}{\omega^8}\left(2k^2+10+\frac{k^2\omega^4}{a^4}\right)-\frac{e^2a^2}{144\pi^3}\cdot$$

$$\cdot\int_0^a \mathrm{d}p\,\exp[p^2/a^2]\,\frac{\sigma_+ + \sigma_-}{\omega_p}\int_0^\infty \mathrm{d}k\,\frac{2\omega+\omega_p}{\omega^3(\omega+\omega_p)^2}\exp[-k^2/a^2]\cdot$$

$$\cdot\left[28\frac{k^6}{a^6}+\left(22+\frac{24}{a^2}\right)\frac{k^4}{a^4}-\left(3-\frac{18}{a^2}\right)\frac{k^2}{a^2}-\frac{2z}{a^2}\right]\approx 12\cdot 10^{-43}\ \text{cm}^3.$$

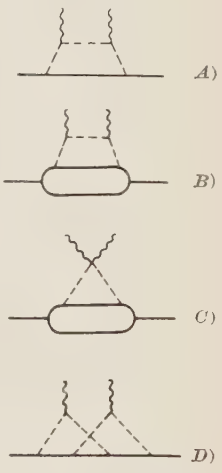

Fig. 1. – ——— the nucleon; – – – the π-meson; ∿∿∿ the electromagnetic field.

Here $\sigma_\pm(p)$ are the total cross-sections for the scattering of $\pi^\pm$-mesons by protons; $\omega_p=\sqrt{1+p^2}$ being the energy of these mesons;

(1) V. S. Barashenkov and B. M. Barbashev: *Nucl. Phys.*, **9**, 426 (1958); D. I. Blokhintsev, V. S. Barashenkov and B. M. Barbashev: *Usp. Fiz. Nauk*, **68**, 417 (1959).

(*) The electric and magnetic polarizabilities of a nucleon are defined by the relation $H=-\frac{1}{2}\alpha E^2-\frac{1}{2}\beta B^2$ in the static limit, where E and B are constant electric and magnetic fields respectively; H is the e^2-proportional energy change of the nucleon due to the fields (1,2).

(2) A. M. Baldin: *Nucl. Phys.*, **18**, 310 (1960).

$\omega = \omega_k$; $f^2 = 0.08$ being the (πN)-coupling constant; $v(k) \equiv \exp[-k^2/2a^2]$ being the form factor of the meson field source; $a = 5.6$. As usual we have set $\hbar = c = \mu_\pi = 1$.

Eq. (1) contains the contributions of the diagrams of the type B, C and D. The matrix element corresponding to the shaded regions of these diagrams is expressed with the aid of the dispersion relations in terms of f^2 and the cross-sections $\sigma_\pm$. (For this it is assumed that the nucleon recoil can be neglected [4]). The α indicated in (1) contains corrections of the order f^4 and higher, which amount to $\Delta\alpha \simeq 2.5 \cdot 10^{-43}$ cm^3. Diagrams of the type D do not contribute to α.

For the magnetic polarizability one obtains the value

$$(2) \qquad \beta = \frac{2e^2}{3\pi} f^2 \int_0^\infty \mathrm{d}k \exp[-k^2/a^2] \left(2k^2 - 6 - \frac{k^2\omega^4}{a^4}\right) \frac{k^4}{\omega^8} +$$

$$+ \frac{16}{27\pi^2} e^2 f^4 \int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}k \exp[-(k^2+p^2)/a^2] k^2 p^2 \left(2 - \frac{k^2}{a^2}\right)\left(2 - \frac{p^2}{a^2}\right) \frac{\omega^2 - \omega_\nu^2}{\omega^3 \cdot \omega_\nu^3 (\omega - \omega_\nu)} +$$

$$+ \frac{e^2}{24\pi^3} \int_0^a \mathrm{d}p \exp[p^2/a^2] \frac{\sigma_+ + \sigma_-}{\omega_p} \int_0^\infty \frac{\mathrm{d}k}{\omega(\omega + \omega_p)} \exp[-k^2/a^2] \cdot$$

$$\cdot \left(\frac{2}{3} \frac{k^6}{a^6} + 9\frac{k^4}{a^4} - 12\frac{k^2}{a^2} + \frac{1}{2} \right) \simeq -0{,}2 \cdot 10^{-43} \text{ cm}^3.$$

In calculating β the diagrams of the type B, C, D are likewise taken into account. The contribution of the diagrams of the type D is compensated almost completely by the contribution of the corrections in B and C. The total contribution of all terms proportional to f^4 and to higher powers of f are $0 < \Delta\beta \leqslant 0.1 \cdot 10^{-43}$ cm^3. (More accurate numerical calculations of these corrections are in preparation).

The values α and β are not very dependent on the shape of $v(k)$.

In the papers [2,3]) experimental values of α are given obtained from the Compton effect on the proton. However the coefficient in the formula for the effective cross-section which has been named in these papers the electric polarizability of the proton is in fact the sum of the two terms:

$$\left[\frac{1}{3} \frac{e^2}{M} \langle r^2 \rangle + \alpha \right],$$

where M is the mass and $\sqrt{\langle r^2 \rangle} = (0.8 \pm 0.04) \cdot 10^{-13}$ cm is the mean square radius of the proton (*).

The consideration of the term containing $\langle r^2 \rangle$ changes noticeably the quantity α obtained from [2,3]).

(3) V. I. GOLDANSKY, O. A. KARPUKHIN, A. V. KUTZENKO and V. V. PAVLOVSKAYA: *Žurn. Éksp. Teor. Fiz.*, **38**, 1695 (1960).

(*) We are indebted to A. M. BALDIN and V. A. PETRUNKIN for discussing this problem (for detail see [4]).

(4) H. MIYAZAWA: *Phys. Rev.*, **104**, 1741 (1956).

α	β	Obtained from
> 4		Photoproduction of mesons on protons [2]
≥ 4		Photoproduction of mesons on protons [7]
	0	Photoproduction of mesons on protons [9]
≤ 12	—	Compton scattering by protons [2]
	$\approx 7.4 - \alpha$ (*)	Consideration of the Compton effect together with the photoptoduction of π-mesons on the proton [2]
5.5 ± 2.3 (*)	2 ± 2 (*)	Consideration of the Compton effect together with the photoproduction of π-mesons on the proton [3]
7.5 ± 4.2	—	Compton scattering by protons [3]
$0 < \alpha < 200$	—	Scattering of slow neutrons by heavy nuclei at small angles [5]
800 ± 350	—	Scattering of slow neutrons by heavy nuclei at small angles [8]

(*) In this case the uncertainties of α and β due to experimental errors in the used values of the photoproduction cross-sections are not indicated.

The values of α and β (in units of 10^{-43} cm^3) are given in the table with the experimental uncertainties indicated. As is seen the theoretical values of α and β are close to those of α and β obtained in [2,3] from the direct analysis of the experimental data.

* * *

We are grateful to D. I. Blokhintsev for discussing the problem.

[5] R. M. Thaler: *Phys. Rev.*, **114**, 827 (1959).
[6] V. A. Petrunkin: *Žurn. Éksp. Teor. Fiz.*, (in print).
[7] L. L. Foldy: (not published, cited by [4]).
[8] Yu. A. Aleksandrov: *Žurn. Éksp. Teor. Fiz.*, **33**, 294 (1957).
[9] L. I. Lapidus and Chou Kuang-Chao: preprint *JINR* D-532, private communication.

Some Properties of the Propagators of a Scalar Field Interacting with Himself.

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(ricevuto il 10 Marzo 1961)

We consider a scalar field $\varphi(x)$ interacting with himself and we wish to give some properties of the generating functional $u[J]$ of the propagators.

As is well known, $u[J]$ can be expressed by means of the ill-defined functional integral in Hilbert-space

$$u[J] = \frac{1}{\mathcal{N}} \int \exp\left[i \int J(\xi)\varphi(\xi)\,\mathrm{d}\xi + i\mathscr{A}_0[\varphi] + i\mathscr{A}'[\varphi] \right] \mathscr{D}\varphi\,, \tag{1}$$

where $\mathscr{A}_0[\varphi]$ is the free-field action and $\mathscr{A}'[\varphi]$ is the interaction [1], $\mathcal{N}$ a normalizing factor such that $u[0]=1$. The n-th order propagator is given by the n-th order derivative of u with respect to J, for $J=0$.

It is important to note that in the following considerations, we shall not need a precise and complete definition of the integral symbol in (1): its translation invariance with respect of its functional argument and the possibility of integrating by parts are the only properties (closely connected together) which we are going to use.

1. – First of all, one can show that $u[J]$ can be expressed as follows [2]:

$$u[J] = \frac{1}{\mathcal{N}} \exp\left[i\mathscr{A}'[-i\,\delta/\delta J(x)] \right] u_0[J]\,, \tag{2}$$

where $u_0[J]$ is the generating functional for free fields

$$u_0[J] = \exp\left[-\frac{1}{4} \int J(\xi)\Delta(\xi-\xi')\,J(\xi') \right], \tag{3}$$

[1] The equation of motion is: $\{\Box - m^2\}\varphi(x) + \delta\mathscr{A}'[\varphi]/\delta\varphi) = 0$.

[2] The numerator and the denominator of the formulas (1) and (2) are defined up to a numerical factor.

where integrals have to be performed over all the variables twice repeated, $\Delta(x)$ is the causal free propagator with $\{\Box - m^2\}\Delta(x) = 2i\,\delta(x)$ and $\mathscr{A}'[-i\,\delta/\delta J(x)]$ is the integro-differential operator obtained by replacing in the functional $\mathscr{A}'[\varphi(x)]$, the function $\varphi(x)$ by $-i\,\delta/\delta J(x)$. Consider, for instance, a non local field theory defined by the interaction

$$\mathscr{A}'[\varphi] = \sum_n \int A(x_n, \ldots x_1)\,\varphi(x_n) \ldots \varphi(x_1)\,, \tag{4}$$

where $A(x_n, \ldots x_1)$ is a function or a distribution (such that one may apply formula (2) to derivative couplings as well). Then one has

$$\mathscr{A}'[-i\,\delta/\delta J(x)] = \sum (-i)^n \int A(x_n, \ldots x_1) \frac{\delta}{\delta J(x_n)} \cdots \frac{\delta}{\delta J(x_1)}\,. \tag{5}$$

All the radiative corrections can be calculated by means of formula (2) and the effects of the interaction are included in its first factor which is the dynamical factor while the kinematical part, independent of the type of interaction, is $u_0[J]$.

2. – Consider now the case where there is a local interaction

$$\mathscr{A}'[\varphi] = g \int \mathscr{L}'(\varphi(x))\,\mathrm{d}x\,, \tag{6}$$

where $\mathscr{L}'$ is a certain function of φ with a Fourier transform with respect of its argument φ:

$$\mathscr{L}'(\varphi) = \int \exp\,[i\lambda\varphi] A(\lambda)\,\mathrm{d}\lambda\,, \tag{7}$$

and $A(\lambda)$ is a function or a distribution. Let us denote the numerator of (1) by $U[J, A\,|g]$ since it is, indeed, a functional of the external source J, the caracterizing function of the interaction A and a function of the coupling constant g. U can be written as follows:

$$U[J, A\,|g] = \int \mathscr{D}\varphi \exp\left\{ i \int J(\xi)\varphi(\xi) + i\mathscr{A}_0[\varphi] + ig \int \exp\,[i\lambda\varphi(x)] A(\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda \right\}. \tag{8}$$

The perturbative expansion of U,

$$U[J, A\,|g] = \sum_n \frac{g^n}{n!} \left| \frac{\partial^n U[J, A\,|g]}{\partial g^n} \right|_{g=0} \tag{9}$$

gives the n-radiative correction of the former functional U as its n-th derivative with respect to g, for $g=0$. By simple algebra, one may prove

$$\left| \frac{\partial^n U[J, A\,|g]}{\partial g^n} \right|_{g=0} = \frac{1}{g^n} \int \left| \frac{\delta^n U[J, A\,|g]}{\delta A(\lambda_n) \ldots \delta A(\lambda_1)} \right|_{A=0} A(\lambda_n) \ldots A(\lambda_1)\,\mathrm{d}\lambda_n \ldots \mathrm{d}\lambda_1 \tag{10}$$

and this formula shows that any radiative correction may be represented as the faltung product of a kinematical factor $[\delta^n U[J, A|g]/\delta A(\lambda_n) \dots \delta A(\lambda_1)]_{A=0}$ independent of the interaction (since A is taken equal to zero!) with the factors $A(\lambda_n) \dots A(\lambda_1)$ which caracterize the interaction. This is a particularly simple way of expressing a property we already noticed in Section **1**, provided that the interaction is local.

On the other hand, this kinematical factor can be brought in a closed form, by means of some calculations based on the translational invariance of (8) with respect to the integration argument φ. One obtains the symmetrical highly singular function of the variables $\lambda_n \dots \lambda_1$:

$$(11) \qquad \frac{1}{g^n} \frac{\delta^n U[J, A|g]}{\delta A(\lambda_n) \dots \delta A(\lambda_1)}\bigg|_{A=0} = (i)^n u_0[J] \int \exp\left[\frac{1}{2} \sum_{p} \lambda_p \int \Delta(x_p - \xi) J(\xi) \right.$$

$$\left. - \frac{1}{4} \sum_{p,q} \lambda_p \lambda_q \Delta(x_p - x_q) \right] \mathrm{d}x_n \dots \mathrm{d}x_1 = u_0[J] W[J/\lambda_n \dots \lambda_1] .$$

Finally, going back to the definition (7) of A, one may bring the n-th radiative correction as given by (9) into the following form:

$$(12) \qquad \frac{\partial^n U[J, A|g]}{\partial g^n}\bigg|_{g=0} = u_0[J] \left\{ \mathscr{L}'\left(-i \frac{\partial}{\partial \lambda_n}\right) \dots \mathscr{L}'\left(-i \frac{\partial}{\partial \lambda_1}\right) W[J/\lambda_n \dots \lambda_1] \right\}_{\lambda=0} .$$

The generating functional $u[J]$ of (1) is now given as the ratio

$$(13) \qquad u[J] = \frac{U[J, A|g]}{U[0, A|g]},$$

of two power series of g. By standard methods one may easily write down the perturbative expansion of $u[J]$: the formulas have been given elsewhere for quantum electrodynamics ([3]).

3. – As we noted in Section **2**, $W[J|\lambda_n \dots \lambda_1]$ is a highly singular function since by formula (11), one sees that we have a sum of distributions: $-\frac{1}{4} \sum \lambda_p \lambda_q \Delta(x_p - x_q)$ as an argument of an exponential. On the light cone, as is well known, the causal A function behaves like the sum of $\delta((x_p - x_q)^2)$ and other singular functions, one sees, therefore, that we need a regularization process in order to transform this W function and its derivatives into distributions. This is an essential step in the renormalization process and will be examined in details later on.

([3]) Applications to quantum electrodynamics are to be found in a paper by A. VISCONTI and H. UMEZAWA to appear in *Compt. Rend. Acad. Sci.*

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A Note on Production Dispersion Relation.

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In the deduction of the dispersion relations for the production processes

$$A+B \to C+D\,,$$

a feature is met, which has no counterpart in the elastic scattering dispersion relation, connected with the fact that the thresholds of the two processes $A+B \to A+B$ and $C+D \to C+D$ are in general different. This problem has been discussed by R. F. Streater and the author [1], but although correct this discussion is unnecessary. We review first the discussion of ref. [1] and at the end we show how the point can be settled by using rotational invariance.

The extension of what is done in the elastic scattering dispersion relation amounts to writing first a dispersion relation for some negative values of

$$\xi = \frac{m_A^2 + m_C^2}{2}\,,$$

with $(m_A^2 - m_C^2)/2$, m_B^2 and m_C^2 fixed at their physical values and the momentum transfer between the B and D particles also fixed, and then performing the analytical continuation up to the physical ξ value. At first, it seems that the absorptive part of the amplitude is not analytic in a neighbourhood S of the real ξ-axis

$$S = \left[\xi = \xi_1 + i\xi_2 : \xi_1 \leqslant \left(\frac{m_A^2 + m_C^2}{2}\right)_{\text{phys}} + \delta\,;\ \xi_2 \leqslant \varepsilon\right],$$

as is the case in the elastic scattering dispersion relation even if both the dispersion relations $A+B \to A+B$, $C+D \to C+D$ can be proved up to a suitably large momentum transfer. In fact, from the application of the Dyson representation, because

[1] A. Minguzzi and R. F. Streater; *Nuovo Cimento* **17**, 946 (1960).

of the absorptive part of the amplitude one has

$$A = \int \frac{\varphi \, \mathrm{d}\alpha \, \mathrm{d}\boldsymbol{u}_i \, \mathrm{d}u_{i0} \, \mathrm{d}\chi_i}{x_1 x_2 - \sqrt{x_1^2 - k_1^2}\,\sqrt{x_2^2 - k_2^2} - k_1 k_2 \cos(\theta - \alpha)}, \qquad i = 1, 2. \tag{1}$$

$$x_1 = \frac{k_1^2 + |\boldsymbol{u}_1^2| + \chi_1^2 - \left(u_{10} + (m_\mathrm{B}^2 - \alpha - \xi)/2W\right)^2}{2|\boldsymbol{u}_1|},$$

$$x_2 = \frac{k_2^2 + |\boldsymbol{u}_2|^2 + \chi_2^2 - \left(u_{20} + (m_\mathrm{D}^2 + \alpha - \xi)/2W\right)^2}{2|\boldsymbol{u}_2|},$$

and

$$k_1 = \frac{1}{2W}\sqrt{(W + m_\mathrm{B})^2 - \xi - \alpha}\,\sqrt{(W - m_\mathrm{B})^2 - \xi - \alpha},$$

$$k_2 = \frac{1}{2W}\sqrt{(W + m_\mathrm{C})^2 - \xi + \alpha}\,\sqrt{(W - m_\mathrm{C})^2 - \xi + \alpha},$$

$$W^2 = (p_\mathrm{A} + p_\mathrm{B})^2.$$

The presence of the product $k_1 k_2$ in the denominator (instead of k^2 as in the elastic scattering case) seems to imply that in the unphysical range the region of analyticity of A in ξ is the strip S around the real ξ-axis minus a neighbourhood around the branch lines of $k_1 k_2$, because one expects in general that there be W-values in which only one of the k_1 and k_2 becomes « imaginary ». (This is not the case in the photo-production amplitude owing to the neglect of the higher order electromagnetic correction). These cuts are harmless if they can be chosen such that a continuous path in S exists linking the unphysical ξ mass-value to its physical value, this path being also contained in the forward tube. In order that A be the absorptive part of the amplitude, it is also required that in the physical range the arguments of k_1 and k_2 be both zero. Letting the branch lines run from the four branch points

$$(W \pm m_\mathrm{B})^2 + \alpha, \qquad (W \pm m_\mathrm{C})^2 - \alpha,$$

up to infinity in the direction of, say, the positive imaginary ξ-axis, allows to fulfil all the requirements for a dispersion relation to hold, provided that the elastic scattering $\mathrm{A} + \mathrm{B} \to \mathrm{A} + \mathrm{B}$, $\mathrm{C} + \mathrm{D} \to \mathrm{C} + \mathrm{D}$ dispersion relation holds for suitably large momentum transfer.

We want to show that no such cuts exist. From Lorentz invariance the weight function depends only on $\cos\alpha$ so that (2) can be cast in the form:

$$A = \int \frac{2\varphi \cdot (x_1 x_2 - \sqrt{x_1^2 - k_1^2}\,\sqrt{x_2^2 - k_2^2} - k_1 k_2 \cos\theta \cos\alpha) \, \mathrm{d}\alpha \, \mathrm{d}\boldsymbol{u}_i \, \mathrm{d}u_{i0} \, \mathrm{d}\chi_i}{(x_1 x_2 - \sqrt{x_1^2 - k_1^2}\,\sqrt{x_2^2 - k_2^2} - k_1 k_2 \cos\theta \cos\alpha)^2 - k_1^2 k_2^2 \sin^2\theta \sin^2\alpha}.$$

and because

$$k_1 k_2 \cos\theta = \left(\frac{W^2 + m_\mathrm{B}^2 - \alpha - \xi}{2W}\right)\left(\frac{W^2 + m_\mathrm{D}^2 + \alpha - \xi}{2W}\right) - \frac{m_\mathrm{B}^2 + m_\mathrm{D}^2}{2} - 2A^2,$$

$$A^2 = -\tfrac{1}{4}(p_\mathrm{B} - p_\mathrm{D})^2,$$

it follows that the singularities of A can arise only from the zeros of the denominator, exactly as it is the case in the proof of the elastic scattering dispersion relation.

LIBRI RICEVUTI E RECENSIONI

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M. Burton, J. S. Kirby-Smith and J. L. Magee – *Comparative Effects of Radiation*. John Wiley & Sons, New York and London, 1960. In ottavo, pp. 426, $ 8.50.

Contiene le relazioni presentate ad una conferenza sugli effetti delle radiazioni infrarosse, visibili, ultraviolette e ionizzanti, e le discussioni che hanno fatto seguito alla esposizione dei vari lavori. La conferenza in oggetto è stata tenuta all'Università San Juan in Puerto Rico, dal 15 al 19 Febbraio 1960 e costituisce un punto di riferimento per il lavoro della « Radiation Research Society » a proseguimento della tradizione cominciata col Simposio di Oberlin e continuata con le Conferenze di Highland Park, che hanno visto riuniti fisici, chimici e biologi intorno ad uno specifico problema di comune interesse.

La prima relazione, svolta da C. P. Swanson, tratta delle differenze riscontrate negli effetti biologici di radiazioni elettromagnetiche di vario tipo: raggi X, ultravioletto ed infrarosso. Il danno provocato dai primi è specialmente genetico; l'ultravioletto ha un effetto fortemente dipendente dalla lunghezza d'onda ed in gran parte reversibile; l'infrarosso sembra essere particolarmente efficace, alle varie lunghezze d'onda, sia come stimolante che come inibitore dello sviluppo vegetale. La seconda relazione, svolta da U. Fano, tratta delle differenze negli effetti riscontrati quando particelle rapide ionizzanti attraversano rispettivamente gas o materiali condensati. Le eccitazioni che nel primo caso

sono prodotte in singoli atomi o molecole, nel secondo caso invece interessano contemporaneamente parecchi atomi. Non si sa ancora come si evolve nel tempo questa eccitazione collettiva e quindi non si è per ora in grado di valutarne l'effetto sui successivi fenomeni chimici e fisici riscontrati. La terza relazione, svolta da S. B. HENDRICKS, tratta dell'effetto delle radiazioni infrarosse sulla crescita delle piante, azione svolta tramite un pigmento fotosensibile (o due varietà di esso). L'effetto è reversibile, le diverse lunghezze d'onda potendo provocare stimolo od inibizione della crescita. La quarta relazione, svolta da C. S. RUPERT, tratta della possibilità di ridurre gli effetti dannosi prodotti da radiazione ultravioletta, mediante successiva irradiazione con luce di maggiore lunghezza d'onda. L'esame del tipo di danneggiamento provocato dall'ultravioletto, in particolare agli acidi nucleinici, permette di farsi un'idea del come la radiazione meno energetica possa riparare il danno iniziale. La quinta relazione, svolta da M. KASHA, cerca di inquadrare in modo unitario la fotochimica molecolare. Viene proposto dall'Autore un nuovo principio fotochimico, secondo cui un quanto di luce assorbito da una molecola poliatomica ha probabilità di venire ripartito fra gli stati più bassi di singoletto e di tripletto della molecola stessa. La sesta relazione, svolta da B. ed A. PULLMANN, tratta degli effetti strutturali delle radiazioni ionizzanti sugli acidi nucleinici. Vengono in particolare esaminate le prove esistenti del differente grado di sensibilità alle radiazioni, presentato dalle basi puriniche rispetto alle basi pirimidiniche contenute nell'ADV. Gli Autori presentano i risultati di alcuni calcoli dell'energia elettronica di risonanza delle due specie molecolari, risultati che sono in accordo con i fatti osservati. Nella settima relazione J. L. MAGEE tenta un'analisi degli eventi che hanno luogo in sistemi molecolari condensati, successivamente al passaggio di una particella ionizzante di alta energia. Hanno luogo contemporaneamente ed in competizione due tipi di fenomeni: diffusione in una zona relativamente vasta dell'energia inizialmente concentrata in un elettrone, e degradazione di questa ad energia di dissociazione molecolare con formazione di stati elettronici metastabili, ed infine degradazione ad energia termica. L'ottava relazione, di C. J. HOCHANADEL, tratta di ciò che avviene nell'acqua, quando viene sottoposta all'azione di radiazioni ionizzanti. Il problema è fisico e chimico ad un tempo; fisico soprattutto nell'analisi dei primi stadi di cessione di energia da parte della radiazione, e della formazione e vita media degli ioni e degli stati eccitati; problema chimico per quel che riguarda le reazioni che successivamente hanno luogo fra i radicali e gli ioni formati con le molecole di acqua e con le sostanze in essa eventualmente disciolte. La nona relazione, svolta da W. H. HAMILL, tratta degli effetti diretti ed indiretti delle radiazioni sulle sostanze allo stato liquido. Lo studio affronta il problema dal punto di vista chimico e cerca di spiegare il comportamento di ioni e radicali liberi. La decima relazione, svolta da M. EBERT, riguarda gli effetti iniziali diretti ed indiretti su sistemi biologici. Due sono le tesi opposte che tentano di spiegare questi effetti delle radiazioni: quella dell'azione diretta della radiazione sul sito molecolare in cui si riscontra il danno, e quello dell'azione indiretta, svolta tramite l'intervento chimico dei radicali liberi inizialmente formati, sui posti « sensibili » della molecola in cui infine si riscontra la modifica. Sembra che le due tesi siano meno opposte di quanto può apparire a prima vista, e che comunque i due tipi di azione possano coesistere in uno stesso sistema. L'undicesima relazione, svolta da A. CHARLESBY, sui recenti progressi nello studio dei polimeri irradiati, riassume sinteticamente gli effetti prodotti dalle radiazioni in queste sostanze, consistenti

nello spezzamento della lunga catena del polimero (degradazione), o nella formazione di legami laterali fra le varie catene, nei punti ionizzati, il che porta a sostanze del tutto nuove, con proprietà molto diverse da quelle del polimero di partenza, e sovente molto interessanti. La dodicesima relazione, svolta da F. HUTCHINSON, tratta degli effetti delle radiazioni su strati monomolecolari, dal punto di vista biologico. L'importanza di questo studio dipende dall'esistenza negli organismi viventi di varie membrane costituite da strati monomolecolari. Gli effetti delle radiazioni vanno interpretati in questo caso in termini di variazioni di permeabilità della membrana. La tradicesima relazione, presentata da T. FORSTEP, tratta del trasporto di eccitazione indotta, da molecola a molecola, o da punto a punto di una stessa macromolecola, senza che ci sia necessariamente contatto diretto fra i due luoghi. Viene discusso lo stato attuale delle teorie del processo e vengono chiarite le relazioni fra le varie teorie esistenti. La quattordicesima ed ultima relazione, svolta da H. KALLMANN, è complementare della precedente, inquantochè tratta dei processi di trasferimento di energia da punto a punto in sistemi solidi o liquidi eccitati mediante bombardamento con radiazioni.

Per dare un giudizio su questo piuttosto insolito volume, è necessario inquadrarlo nel particolare momento attuale, in cui i recenti rapidi aumenti nelle nostre conoscenze degli effetti fisici e degli effetti chimici e biologici delle radiazioni, rendono sempre più necessari gli scambi d'idee e le discussioni comuni fra gli specialisti delle tre scienze interessate, per il maggior profitto ed il più opportuno indirizzo degli studi di ognuno.

Il volume anzi trae il suo motivo di interesse più generale, proprio dal fatto di risultare da uno sforzo comune di studiosi di discipline diverse, intorno al comune problema.

In questa veste e sotto questa specie lo suggeriamo alla lettura anche di quanti non fossero direttamente interessati al suo argomento specifico.

F. GAETA

E. M. PUGH and E. W. PUGH: *Principles of Electricity and Magnetism*. Addison-Wesley, Reading - London, 1960; $ 8.75.

Questo libro si propone di approfondire la conoscenza dei principi dell'elettricità e del magnetismo per studenti che indirizzano la loro attività verso i diversi rami della tecnica, principalmente, si può ritenere, verso l'ingegneria e la fisica applicata. Esso presuppone una conoscenza della fisica elementare e dei fondamenti dell'algebra e del calcolo infinitesimale, sino alle equazioni a derivate parziali.

Il primo capitolo è dedicato ai sistemi di unità di misura in uso per l'elettromagnetismo. Seguono tre capitoli sui fondamenti dell'elettrostatica, i quali servono a richiamare insieme le principali nozioni matematiche occorrenti, ed in particolare a introdurre i concetti e i metodi dell'analisi vettoriale, di cui è fatto poi uso sistematico nel libro. Gli A. rilevano come sia vantaggioso ai fini didattici presentare i principi della matematica insieme con le applicazioni fisiche di essi. Pur trovandoci d'accordo, in linea generale, su questo punto di vista, solleviamo qualche dubbio sulla opportunità di presentare la legge di Biot e Savart, come applicazione del prodotto vettoriale, ed il teorema di Stokes, come applicazione dell'operatore *rot*, in capitoli di elettrostatica, nel timore che questo possa generare qualche confusione d'idee.

L'inconveniente, se tale lo si vuol considerare, che il titoli dei capitoli non corrispondano sempre strettamente al contenuto, si riscontra anche più avanti. Al capitolo sesto sulle « Correnti continue »

segue il settimo sulla « Induzione elettromagnetica », mentre l'ottavo soltanto è dedicato ai « Campi magnetici delle correnti elettriche »: ma le definizioni relative a questi campi si trovano nel capitolo settimo. È probabile, del resto, che l'allievo che fa uno studio sistematico del libro non abbia a risentire di questi inconvenienti; mentre egli troverá utili i frequenti ravvicinamenti fra argomenti diversi.

A parte questi rilievi non essenziali, ed altri che si potrebbero fare circa qualche squilibrio nella trattazione, il libro ci sembra utile a quei fini didattici che esso si propone. L'impostazione generale e la scelta della materia sono quelle tradizionali. Il libro si conclude con un capitolo abbastanza esteso sui circuiti a corrente alternata, ed uno sulla radiazione elettromagnetica, che contiene la deduzione delle leggi della riflessione e rifrazione per un'onda piana e trattazioni sommarie dei problemi del dipolo oscillante e delle guide d'onda.

A. Rostagni

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